APPLICATIONS OF THE UNIVERSAL COEFFICIENT THEOREM FOR CONNECTIVE K-THEORY

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ABSTRACT

In this thesis we compute the $BP\langle 1\rangle$ cohomology of $BP\langle n\rangle$. This is done by contrasting the two different associated gradeds for $BP\langle 1\rangle^*BP\langle n\rangle$ provided by the Adams spectral sequence and the universal coefficient spectral sequence to prove that both spectral sequences collapse to their E_2 terms. The same technique is used to provide a splitting of $BP\langle 1\rangle \wedge BP\langle n\rangle$ and to prove a proposition crucial to the proof that $\mathrm{holim}_k\ P_{-k} \wedge BP\langle 2\rangle$ splits as a product of suspensions of $BP\langle 1\rangle$'s. The results for $BP\langle 1\rangle^*BP\langle 2\rangle$ are used to construct generalized Brown-Gitler spectra over $BP\langle 2\rangle$.

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CHAPTER 1. INTRODUCTION

In [Ba] N. Baas, using an idea of D. Sullivan [Su1] used cobordism of manifolds with singularities to construct, (for each prime p and nonnegative integer n) certain p-local spectra BP<n>. They were constructed to have $\pi_*BP\langle n\rangle \cong Z_{(p)}[v_1,\ldots v_n]$ and $H^*(BP\langle n\rangle;F_p) \cong A\otimes_{E[Q_0,\ldots Q_n]}F_p$, where the Q_i are the Milnor elements in the Steenrod algebra A. These isomorphisms come from compatible maps $BP \to BP\langle n\rangle \to HF_p$ which give $\pi_*BP\langle n\rangle$ as a quotient module of π_*BP and $HF_p{}^*BP\langle n\rangle$ as a quotient module of A. With just this much knowledge of BP<n>, much can be done to compute $BP\langle n\rangle_*X$ and $BP\langle n\rangle^*X$ via the Adams spectral sequences

$$\mathrm{Ext}_{E[Q_0,Q_1,\ldots Q_n]}(H^*X,F_p) \Longrightarrow BP\langle n\rangle_*X$$

$$\mathrm{Ext}_{E[Q_0,Q_1,\ldots Q_n]}(F_p,H^*X) \Longrightarrow BP\langle n\rangle^*X.$$

The next thing one would wish to know about the BP<n> is whether or not they are ring spectra. This took a long time to settle. In 1976, in [SY], Shimada and Yagita proved the existence of an external multiplication, $BP\langle n\rangle_*X\otimes BP\langle n\rangle_*Y \to BP\langle n\rangle_*X\wedge Y$ which may not be commutative when p=2, but this is not sufficient to establish that BP<n> is a ring spectrum. In 1979 in [Mo], Morava, using the results of [SY], settled the question for primes larger than three. Finally in 1984 in [Rav1],Ravenel, using results of Yosimura [Y], settled the question for all primes, namely $BP\langle n\rangle$ is an associative and commutative (if $p>2$) ring spectrum.

Since $H^*BP\langle n\rangle$ splits as a module over $E[Q_0,\dots,Q_m] = E^m$ (see [LM]) this leads one to suspect that this splitting can be realized by a splitting of $BP\langle n\rangle\wedge BP\langle m\rangle$.

In [Mah1], Mahowald gave a splitting of $bo\wedge bo$. In [K], Kane used Mahowald's methods to split $BP\langle 1\rangle\wedge BP\langle 1\rangle$ for p an odd prime. Their splittings used spaces $BP\langle 0\rangle^k$ which they constructed and which later ([GJM]) came to be considered generalized Brown-Gitler spectra over $BP\langle 0\rangle$ since $H^*BP\langle 0\rangle^k = A/(A\cdot\{\beta\}\cup A\cdot\{\chi\beta^{\varepsilon}P^j\,|\,\varepsilon=0 \text{ or } 1,\ j>k\})$ while $H^*BP\langle 0\rangle = A/A\cdot\{\beta\}$ and the cohomology of the original Brown-Gitler spectra were $H^*B_k = A/A\cdot\{\chi(\beta^{\varepsilon}P^j)\,|\,\varepsilon=0 \text{ or } 1,\ j>k\}$. Mahowald used the $bo\wedge bo$ splitting to get information about v_1 periodicity of spheres out of his bo based analogue to the Adams spectral sequence. Kane used it to construct operations with which he proved that for p odd and $HZ_{(p)}{}^*X$ torsion free, if P^{p^s} acts trivially on $HF_p{}^*X$ then so does P^n when $n\geq p^s$. Lellmann in [Le] used the splitting to determine $BP\langle 1\rangle^*BP\langle 1\rangle$.

We reverse the order of Lellmann's paper and first compute $BP\langle 1\rangle^*BP\langle n\rangle$ in chapter five and then provide a splitting of $BP\langle 1\rangle\wedge BP\langle n\rangle$ in chapter six. As preparation for these results the second chapter of this thesis gives a splitting of $H^*BP\langle m\rangle\wedge BP\langle n\rangle$ as a module over the Steenrod algebra and a splitting of

$BP\langle 0\rangle\wedge BP\langle n\rangle$ as background for the kinds of things that should happen in the general case. The third chapter reviews the computation of the E^1 module structure of $H^*BP\langle n\rangle$ and Ext_{E^1} computations culminating in the computation of $BP\langle 1\rangle_*BP\langle n\rangle$. In the fourth chapter, the computation of $BP\langle 1\rangle^*BP\langle n\rangle$ takes advantage of the two different associated gradeds for $[BP\langle n\rangle,BP\langle 1\rangle]$ provided by the Adams spectral sequence and the $BP\langle 1\rangle$ homology universal coefficient spectral sequence,

$$\mathrm{Ext}^{s,t}_{BP\langle 1\rangle_*}(BP\langle 1\rangle_*BP\langle n\rangle,BP\langle 1\rangle_*)\Longrightarrow [BP\langle n\rangle,BP\langle 1\rangle]_{t-s}.$$

The existence of such a spectral sequence was recently proved in [R3] where Robinson proves the existence of such spectral sequences for all A_∞ ring spectra which Adams [A2] way back in 1969 conjectured should exist. The fifth chapter uses the same techniques to provide a splitting of $BP\langle 1\rangle\wedge BP\langle n\rangle$ as a wedge of suspensions of $BP\langle 0\rangle^k\wedge BP\langle 1\rangle$ for various k. The sixth chapter uses the computation of $BP\langle 1\rangle^*BP\langle 2\rangle$ to construct $BP\langle 2\rangle^k$, the generalized Brown-Gitler spectra over $BP\langle 2\rangle$. The strategy of the proof is the same as the one in [GJM] with just a few significant alterations needed. The seventh and final chapter presents an application of the techniques of chapter four to prove that

$$\mathrm{holim}_k P_{-k}\wedge BP\langle 2\rangle \simeq \Pi_{i\in Z}\Sigma^{qi-1}BP\langle 1\rangle^\wedge.$$

This represents joint work with D. Davis, D. Johnson, M. Mahowald and S. Wegmann. The critical point here is that one gets a spectrum C_k with $H^*C_k \simeq \oplus_{i\geq k}\Sigma^{2i-1}H^*BP\langle 1\rangle$ and one needs to show that $C_k \simeq \vee_{i\geq k}\Sigma^{2i-1}BP\langle 1\rangle$. It suffices to show that the Adams spectral sequence for $[C_k, BP\langle 1\rangle]$ collapses which follows from contrasting it with the universal coefficient spectral sequence.

In many places throughout this thesis we are working with spectra localized or completed at a prime p. Sometimes the case p=2 should have been dealt with as a special case but was not because a simple substitution will give the correct statement and proof for the case p=2 from the p odd case. One of the primary examples occuring in chapters 2-4 is the dual of the Steenrod algebra. When p is odd this is $F_p[\xi_1,\xi_2,\ldots]\otimes E[\tau_0,\tau_1,\ldots]$ while for p=2 it is $F_2[\zeta_1,\zeta_2,\ldots]$. The substitution of $\zeta_i{}^2$ for ξ_i and ζ_{i+1} for τ_i, which will give the correct conversion. Another important example occurs in chapter five. There to get the correct 2-primary statements substitute Sq^{2k} for P^k and Sq^1 for β.

One word about the grading of modules. Cohomology modules should always be thought of as graded by codegree. Thus $\mathrm{Hom}^t(M,N)$ means homomorphisms which increase degree by t or decrease codegree by t. One should be careful about the fact that a module over the coefficients E_* of any homology theory can be thought of both as a cohomology module and a homology module with the corresponding change from degree to codegree.

We work in any category of spectra satisfying the axioms of [M] such as the category of CW spectra of [A1]. The only place in which unstable spaces appear is in chapter six in constructing adaptive complexes and it is possible that elsewhere the word space may have been used when talking about a spectrum. All homology theories are reduced and ordinary homology is with mod p coefficients unless otherwise specified.

In chapter six I need the rings $E[Q_s,Q_{s+1},\ldots Q_n]$ and call them E^n_s following [JW]. Thus throughout this thesis I use E^n for $E[Q_0,\ldots Q_n]$ which differs from the more common notation of E_n. Here and in [JW] E_n means $E[Q_n,Q_{n+1},\ldots]$.

CHAPTER 2. TOOLS FOR MAKING BP<n> COMPUTATIONS

In this chapter we introduce the Adams spectral sequence used to compute $BP\langle n\rangle_* BP\langle n+k\rangle$ and $BP\langle n\rangle^* BP\langle n+k\rangle$. To facilitate computation we are led to give a splitting of $H^* BP\langle n\rangle \wedge BP\langle n+k\rangle$. We finish by doing the computations for the first case, $BP\langle 0\rangle_* BP\langle k\rangle$ and $BP\langle 0\rangle^* BP\langle k\rangle$. These results are not new and we present them here simply to put the later results into context.

Theorem 2.1 ([4.5;W] or [Ch.16;M])

If X and Y are spectra of finite type then the classical mod-p Adams spectral sequence converges strongly to $[X,Y_p^\wedge]_*$.

Here $Y_p^\wedge = Y \wedge S_p^\wedge$ means the p-completion of Y. Strong convergence is used in the sense of Boardman [B] and here it means that the usual Adams filtration of $[X,Y_p^\wedge]$ is Hausdorff and complete. So from theorem 2.1 we can be assured that when X is connective and of finite type, two different spectral sequences converge strongly.

$$\mathrm{Ext}_A(H^* BP\langle n\rangle \wedge X, Z/p) \Longrightarrow [S, BP\langle n\rangle \wedge X \wedge S_p^\wedge]$$

$$\mathrm{Ext}_A(H^* BP\langle n\rangle, H^* X) \Longrightarrow [X, BP\langle n\rangle \wedge S_p^\wedge]$$

Since we will always use these two spectral sequences to compute $BP\langle n\rangle$ homology and cohomology, from now on we will write $BP\langle n\rangle$ for the p-completion of $BP\langle n\rangle$.

Now [p338;A1] implies that

$$H^*BP\langle n\rangle\wedge X \cong (A\otimes_{E^n} Z/p)\otimes H^*X$$

$$\cong A\otimes_{E^n} H^*X$$

where these are isomorphisms as left A modules with A acting by the diagonal action on the first two modules and by the left action on the third module. From this a change of rings isomorphism applied to the spectral sequences gives

$$\mathrm{Ext}_{E^n}(H^*X,F_p) \Longrightarrow [S,BP\langle n\rangle\wedge X] \qquad (A)$$

$$\mathrm{Ext}_{E^n}(F_p,H^*X) \Longrightarrow [X,BP\langle n\rangle] \qquad (B)$$

At this stage we should say something about product structure. Since $BP\langle n\rangle$ is a ring spectrum with structure map μ, the map

$\alpha:BP\langle n\rangle\wedge BP\langle n\rangle\wedge X \xrightarrow{\mu\wedge Id_X} BP\langle n\rangle\wedge X$ gives $BP\langle n\rangle_*X$ a module sructure over $BP\langle n\rangle_*$. We would like the Adams spectral sequences to tell us something about this structure. Theorem 2.3.3 of Ravenel's book [Rav], states that the map α induces a natural pairing

$E_r^{**}(BP\langle n\rangle)\otimes E_r^{**}(BP\langle n\rangle\wedge X) \to E_r^{**}(BP\langle n\rangle\wedge X)$ for $r\geq 2$ such that the differential is a derivation and the pairing on E_∞ corresponds to $\alpha_*:BP\langle n\rangle_*\otimes BP\langle n\rangle_*X \to BP\langle n\rangle_*X$. At the E_2 level the pairing is the Yoneda product

$$\mathrm{Ext}_{E^n}(H^*X,F_p)\otimes \mathrm{Ext}_{E^n}(F_p,F_p) \to \mathrm{Ext}_{E^n}(H^*X,F_p).$$

The pairing for $BP\langle n\rangle^*X$ works similarly.

There is a splitting of $H^*BP\langle m\rangle$ over $E=E[Q_0,Q_1,\ldots]$ which will facilitate the computation of the Ext groups needed to compute $BP\langle n\rangle_*BP\langle m\rangle$ and $BP\langle n\rangle^*BP\langle m\rangle$ via these spectral sequences.

In order to see this splitting it turns out to be easier to proceed with a left action of E on the dual of $H^*BP\langle m\rangle$. Recall that the canonical antiautomorphism of the Steenrod algebra converts the left action of E on $A/(AQ_0+\ldots+AQ_m)$ to the right action of E on $A/(Q_0A+\ldots+Q_mA)$. Now the right action of E dualizes to the left action of E on the dual of $A/(Q_0A+\ldots+Q_m)$. This dual is isomorphic to

$$F_p[\xi_1,\xi_2,\ldots]\otimes E[\tau_{m+1},\tau_{m+2},\ldots] \cong \chi H_*BP\langle m\rangle$$

as a sub Hopf algebra of A_*. The action of E on this is given by

$$Q_i\xi_j = 0$$

$$Q_i\tau_j = \begin{cases} 0 & j<i \\ \xi_{j-i}^{p^i} & j\geq i \end{cases}$$

$$Q_i(a\cdot b) = (Q_ia)b+(-1)^{|a|}a(Q_ib)$$

(see [ABP]). Following [K] and [Mahl] we filter $F_p[\xi_1,\xi_2,\ldots]\otimes E[\tau_{m+1},\tau_{m+2},\ldots]$ by setting

$$\mathrm{wt}\ \xi_i = p^i = \mathrm{wt}\ \tau_i$$

$$\mathrm{wt}(ab) = \mathrm{wt}\ a + \mathrm{wt}\ b.$$

Let $M_{m,j}$ be the elements of weight j. Since the Q_i preserve weight

$$\chi H_*BP\langle m\rangle \cong \oplus_{j\geq 0}M_{m,j}$$

as modules over E. Dually

$$\chi H^*BP\langle m\rangle \cong \oplus_{j\geq 0}M_{m,j}^*$$

as modules over E. One should notice that these modules split even further when one works over subalgebras of E.

To illustrate the use of these techniques we will run through the calculations of $BP\langle 0\rangle_* BP\langle k\rangle$ and $BP\langle 0\rangle^* BP\langle k\rangle$. These calculations are very simple because the only indecomposable modules over $E^0 = E[Q_0]$ are E^0 and Z/p. Since $Q_0 Q_0 = 0$, Q_0 acts as a differential on all modules over E^0 and we can define

$$H(M;Q_0) = \ker Q_0/\operatorname{Im} Q_0.$$

If $H(M;Q_0) \cong \oplus_i \Sigma^{n_i} Z/p$ then M splits as $\oplus_i \Sigma^{n_i} Z/p \oplus V \oplus E^0$ for some graded F_p vector space V.

Let us determine the Q_0 homology of the module $F_p[\xi_1,\xi_2,\ldots]\otimes E[\tau_{m+1},\tau_{m+2},\ldots]$. Under the differential Q_0 it can be expressed as a tensor product of chain complexes. The first factor is the polynomial algebra $F_p[\xi_1,\xi_2,\ldots\xi_m]$ with the zero boundary. The r^{th} factor for $r\geq 2$ has a base of monomials

$$\tau_{r+m-1}\xi^i_{r+m-1} \text{ and } \xi^i_{r+m-1} \text{ for } i=0,1,2,\ldots$$

with boundary

$$\tau_{r+m-1}\xi^i_{r+m-1} \to \xi^{i+1}_{r+m-1}.$$

Hence the r^{th} factor has homology F_p and the Q_0 homology of $F_p[\xi_1,\xi_2,\ldots]\otimes E[\tau_{m+1},\tau_{m+2},\ldots]$ is $F_p[\xi_1,\xi_2,\ldots\xi_m]$.

Let $K_m = \{I=(i_1,\ldots i_m) \mid i_k \text{ is a nonnegative integer for } 1\leq k\leq m\}$.

The degree of $\xi^I = \xi_1^{i_1}\xi_2^{i_2}\ldots\xi_m^{i_m}$ is

$$d(I) = 2\Sigma_{j=1}^{m} i_j(p^j-1).$$

Hence as a module over E^0

$$H^*BP\langle m\rangle \cong \oplus_{I\in K_m} \Sigma^{d(I)} Z/p \oplus (V\otimes E^0)$$

for some graded F_p vector space $V = \oplus_{j\in J}\Sigma^{n_j} Z/p$.

From a minimal resolution of Z/p as an E^0 module we can compute that

$$\mathrm{Ext}^{s,t}_{E^0}(Z/p,Z/p) \begin{cases} Z/p & \text{if } s=t \\ 0 & \text{if } s\neq t \end{cases}$$

$$\mathrm{Ext}^{s,t}_{E^0}(Z/p,E^0) \begin{cases} Z/p & \text{if } s=0, t=-1 \\ 0 & \text{else} \end{cases}$$

From a change of rings we can compute that

$$\mathrm{Ext}^{s,t}_{E^0}(E^0,Z/p) \begin{cases} Z/p & \text{if } s=t=0 \\ 0 & \text{else} \end{cases}$$

$$\mathrm{Ext}^{s,t}_{E^0}(E^0,E^0) \begin{cases} Z/p & \text{if } s=0 \text{ and } t=0 \text{ or } -1 \\ 0 & \text{else} \end{cases}$$

The Yoneda product for Ext makes $\mathrm{Ext}^{s,t}_{E^0}(Z/p,Z/p)$ into a ring which can readily be determined to be $F_p[q_o]$ where q_0 is a generator of $\mathrm{Ext}^{1,1}_{E^0}(Z/p,Z/p)$. We now have sufficient information to realize that

$$\mathrm{Ext}^{**}_{E^0}(H^*BP\langle m\rangle,Z/p) \cong \oplus_{I\in K_m} \Sigma^{d(I)} F_p[q_0] \oplus V$$

where V is a graded F_p vector space concentrated in filtration zero. (Notice that the inclusion $F_p \to F_p[x_1,\ldots,x_n]$ is a ring homomorphism which gives any graded F_p vector space the structure of a graded $F_p[x_1,\ldots,x_n]$ module, i.e. the x_i's act trivially, a fact used continually throughout this thesis). Hence all the groups in the E_2 term of the Adams spectral sequence

$$\mathrm{Ext}^{s,t}_{E^0}(H^*BP\langle m\rangle, Z/p) \Longrightarrow BP\langle 0\rangle_{t-s}BP\langle m\rangle$$

are either of infinite order with respect to q_0 multiplication lying in even stems, or else they are q_0 torsion lying in filtration zero. Since the differentials decrease stem by one and commute with q_0 multiplication, all of them must be zero. Thus this spectral sequence collapses to its E_2 term.

Knowledge of the module structure of the Ext groups over $F_p[q_0]$ allows us to solve the extension problems and we get

$$BP\langle 0\rangle_* BP\langle m\rangle \approx \oplus_{I\in K_m} \Sigma^{d(I)} Z_p^\wedge \oplus V.$$

To compute $BP\langle 0\rangle^* BP\langle m\rangle$ we need to use

$$\mathrm{Ext}^{s,t}_{E^0}(F_p, H^*BP\langle m\rangle) \Longrightarrow [BP\langle m\rangle, BP\langle 0\rangle]_{t-s}$$

where by our previous work

$$\mathrm{Ext}^{s,t}_{E^0}(F_p, H^*BP\langle m\rangle) \approx \oplus_{I\in K_m} \Sigma^{-d(I)} F_p[q_0] \oplus \Sigma^{-1} V^*$$

where V^* is the F_p dual of V. Again the torsion free part of this module is concentrated in even stems and the torsion part is concentrated in filtration zero, so all differentials are zero. Hence

$$BP\langle 0\rangle^* BP\langle m\rangle \approx \oplus_{I\in K_m} \Sigma^{d(I)} Z_p^\wedge \oplus \Sigma^1 V.$$

Since $\pi_* BP\langle 0\rangle \wedge BP\langle m\rangle$ splits as a direct sum of copies of $Z_p^\wedge$ and Z/p, we might hope to find a splitting of $BP\langle 0\rangle \wedge BP\langle m\rangle$ as a wedge of spectra $BP\langle 0\rangle$ and $BP\langle 0\rangle \wedge M$ where $\pi_* BP\langle 0\rangle \wedge M = F_p$. The M we want is the Moore spectrum for F_p which we denote M_p.

For any $I' \in K_m$ we have the surjection

$$\oplus_{I \in K_m} \Sigma^{d(I)} Z/p \oplus V \otimes E^0 \to \Sigma^{d(I')} Z/p.$$

Applying $A \otimes_{E^0}$ ___ gives

$$A \otimes_{E^0} (\oplus_{I \in K_m} \Sigma^{d(I)} Z/p \oplus V \otimes E^0) \to A \otimes_{E^0} \Sigma^{d(I')} Z/p.$$

To realize this map take the map $S^{d(I')} \to BP\langle 0\rangle \wedge BP\langle m\rangle$ coming from $Ext^{0,d(I')}_{E^0}(\Sigma^{d(I')} Z/p, Z/p)$ which is a summand of the Ext groups for computing $\pi_* BP\langle 0\rangle \wedge BP\langle m\rangle$. Smashing both sides with $BP\langle 0\rangle$ and using the ring structure of $BP\langle 0\rangle$ gives a map

$$\Sigma^{d(I')} BP\langle 0\rangle \to BP\langle 0\rangle \wedge BP\langle m\rangle$$

which realizes the given cohomology homomorphism.

Similarly consider the surjection

$$\oplus_{I \in K_m} \Sigma^{d(I)} Z/p \oplus V \otimes E^0 \to \Sigma^n Z/p \otimes E^0$$

where $\Sigma^n Z/p$ is one of the summands in V. Now we have the Adams spectral sequence

$$Ext^{s,t}_{E^0}(H^* BP\langle m\rangle, \Sigma^n E^0) \Longrightarrow [\Sigma^n M_p, BP\langle 0\rangle \wedge BP\langle m\rangle]_{t-s}.$$

All of these Ext groups are in filtration zero so there are no non-trivial differentials. Hence the generator of the Z/p in bidegree (0,0) in $Ext_{E^0}(\Sigma^n E^0, \Sigma^n E^0)$ gives an element of

$Ext_{E^0}(H^* BP\langle m\rangle, H^* M_p)$ and thus a map $\Sigma^n M_p \to BP\langle 0\rangle \wedge BP\langle m\rangle$. Again smashing with $BP\langle 0\rangle$ and using the ring structure of $BP\langle 0\rangle$ gives a map

$$\Sigma^n M_p \wedge BP\langle 0\rangle \to BP\langle 0\rangle \wedge BP\langle m\rangle$$

which realizes the given map in cohomology.

Wedging all these maps together gives a map

$$(\bigvee_{I\in K_m} \Sigma^{d(I)} BP\langle 0\rangle) \vee \bigvee_{j\in J} \Sigma^{n_j} M_p \wedge BP\langle 0\rangle \to BP\langle 0\rangle \wedge BP\langle m\rangle$$

which is an isomorphism in cohomology.

Remarks Notice that success in the computation of $BP\langle 0\rangle^* BP\langle m\rangle$ and $BP\langle 0\rangle_* BP\langle m\rangle$ was dependent on two factors. First modules over E^0 are readily determined and all the relevant Ext groups are easy to compute. Secondly the relevant spectral sequences all collapse because all the elements of positive filtration are torsion free and lie in even stems. Now for the case of $BP\langle 1\rangle_* BP\langle m\rangle$ and $BP\langle 1\rangle^* BP\langle m\rangle$ we need to work with modules over E^1. In [A1] or [M] we can find a complete determination of modules over E^1 and the techniques to compute the relevant Ext groups. So the difficulty comes in trying to show that the relevant Adams spectral sequences collapse. The "everything in even stems" argument works for $BP\langle 1\rangle_* BP\langle m\rangle$ but does not work for $BP\langle 1\rangle^* BP\langle m\rangle$ or for the spectral sequences needed to get the splitting maps. This is where the universal coefficient spectral sequence for $BP\langle 1\rangle$ comes into play.

For the higher cases $BP\langle n\rangle_* BP\langle m\rangle$ and $BP\langle n\rangle^* BP\langle m\rangle$ when $n\geq 2$ we have neither a complete determination of modules over E^n nor the ability to compute the Ext groups so we do not even reach the stage of worrying about the collapsing of the spectral sequences. If we did then we would add to our difficulties the fact that it is not known whether the relevant universal coefficient spectral sequence exists, since it is not known whether BP<n> is an A_∞ ring spectrum for $n \geq 2$.

CHAPTER 3. THE BP<1> HOMOLOGY OF BP<n>

Since the Adams spectral sequence used for this computation will collapse to the E_2 term, the main task of this chapter is the computation of the Ext groups. To perform this calculation we use the method and notation developed in [AP] and [A1], (also see [M]).

We start by determining the E^1 module structure of $H^*BP\langle n\rangle$. As before it is easier to work in the dual,

$$(A/E^n\cdot A)_* = F_p[\xi_1,\xi_2,\ldots]\otimes E[\tau_{n+1},\tau_{n+2},\ldots]$$

with the action of $E[Q_0,Q_1]$ given by

$$Q_0\tau_i = \xi_i \qquad\qquad Q_0\xi_i = 0$$

$$Q_1\tau_i = \xi_{i-1}^p \qquad\qquad Q_1\xi_i = 0$$

$$Q_i(a\cdot b) = (Q_i a)\cdot b + (-1)^{|a|}aQ_i b \qquad \text{for } i=0,1.$$

Under the first boundary, Q_0, the complex has homology $F_p[\xi_1,\xi_2,\ldots\xi_n]$ as discovered in the previous chapter.

Under the second boundary the complex is also a tensor product of chain complexes. This time the first factor is $F_p[\xi_1,\ldots,\xi_{n-1}]$ with zero boundary and the r^{th} factor for $r\geq 2$ has a base of monomials

$$\tau_{r+n-1}\xi_{r+n-2}^i \text{ and } \xi_{r+n-2}^i \qquad\qquad (i=0,1,2\ldots)$$

with boundary

$$\tau_{r+n-1}\xi^{i}_{r+n-2} \to \xi^{i+p}_{r+n-2}, \quad \xi^{i}_{r+n-2} \to 0 \qquad (i=0,1,\ldots)$$

The homology of this factor is $F_p[\xi_{r+n-2}]/\xi^p_{r+n-2}$. Thus the Q_1 homology of $(A/\bar{E}^n\cdot A)_*$ is a truncated algebra generated by $\xi_1,\xi_2,\ldots$ with relations $\xi_n^p=0$, $\xi_{n+1}^p=0,\ldots$.

Armed with the knowledge of the Q_0 and Q_1 homology of $(A/E^n\cdot A)_*$ we can use the ideas of [AP] to determine the $E[Q_0,Q_1]$ module structure of $A/A\cdot\bar{E}^n \approx H^*BP\langle n\rangle$. For any $x\in(A/\bar{E}^n\cdot A)_*$ write Dx for the dual element with respect to the polynomial basis for $(A/\bar{E}^n A)_*$ as an F_p vector space. Since $Q_ix=y$ implies that $Dy\cdot Q_i=Dx$ we see that x generates Q_i homology if and only if Dx generates Q_i homology.

For $m\geq 0$, let $L(m)$ be the module over E^1 given by

generators: a_i, $0\leq i\leq m$ $\qquad |a_i| = qi$ where $q = 2(p-1)$

relations: $Q_1a_i = Q_0a_{i+1}$ $\quad 0\leq i<m$

$Q_0a_0 = 0 = Q_1a_m$.

For $m<0$ let $L(m)$ be given by

generators: a_i, $m\leq i\leq -1$ $\qquad |a_i| = qi-1$

relations: $Q_1a_i = Q_0a_{i+1}$ $\quad m\leq i\leq -2$.

If ν_p is the function defined by

$\nu_p(b)=m$ when $p^m|b$ and $p^{m+1}\nmid b$

and α_p is the function that assigns to b the sum of the coefficients of the p-ary expansion of b then

$$\tau_{r+n-1}\xi^{i}_{r+n-2} \to \xi^{i+p}_{r+n-2},\quad \xi^{i}_{r+n-2} \to 0 \qquad (i=0,1,\ldots)$$

The homology of this factor is $F_p[\xi_{r+n-2}]/\xi^p_{r+n-2}$. Thus the Q_1 homology of $(A/\bar{E}^n\cdot A)_*$ is a truncated algebra generated by $\xi_1,\xi_2,\ldots$ with relations $\xi^p_n=0$, $\xi^p_{n+1}=0,\ldots$.

Armed with the knowledge of the Q_0 and Q_1 homology of $(A/E^n\cdot A)_*$ we can use the ideas of [AP] to determine the $E[Q_0,Q_1]$ module structure of $A/A\cdot\bar{E}^n \cong H^*BP\langle n\rangle$. For any $x\epsilon(A/\bar{E}^n\cdot A)_*$ write Dx for the dual element with respect to the polynomial basis for $(A/\bar{E}^n A)_*$ as an F_p vector space. Since $Q_i x=y$ implies that $Dy\cdot Q_i=Dx$ we see that x generates Q_i homology if and only if Dx generates Q_i homology.

For $m\geq 0$, let $L(m)$ be the module over E^1 given by

generators: a_i, $0\leq i\leq m$ $\qquad |a_i| = qi$ where $q = 2(p-1)$

relations: $Q_1a_i = Q_0a_{i+1}$ $\quad 0\leq i<m$

$Q_0a_0 = 0 = Q_1a_m$.

For $m<0$ let $L(m)$ be given by

generators: a_i, $m\leq i\leq -1$ $\qquad |a_i| = qi-1$

relations: $Q_1a_i = Q_0a_{i+1}$ $\quad m\leq i\leq -2$.

If ν_p is the function defined by

$\nu_p(b)=m$ when $p^m|b$ and $p^{m+1}\nmid b$

and α_p is the function that assigns to b the sum of the coefficients of the p-ary expansion of b then

$$\nu_p(b!) = \frac{b-\alpha_p(b)}{p-1} = \Sigma^s_{i=0}\ b_i(p^{i-1}+\ldots+p+1)$$

when $b = b_0+b_1p+\ldots+b_sp^s$ is the p-ary expansion of b. Define the functions β_i and γ_i by

$$\beta_i(b) = \Sigma_{j=i}^{s}\ b_jp^{j-i}$$

$$\gamma_i(b) = \Sigma_{k=1}^{i}\ \beta_k(b),\quad \gamma_0(b) = 0.$$

Notice that $\gamma_s(b) = \nu_p(b!)$ and that $\gamma_i(b)$ is a strictly increasing function of i.

We define a map

$$h:A/\bar{E}^n\cdot A \to \oplus_{I\epsilon K_n}\ \Sigma^{d(I)}L(\nu_p(i_n!)) \approx \Pi_{I\epsilon K_n}\ \Sigma^{d(I)}L(\nu_p(i_n!))$$

by giving the component

$$h_I:A/\bar{E}^n\cdot A \to \Sigma^{d(I)}L(\nu_p(i_n!)).$$

If $i_n=0$ the map is simply

$$D(\xi^I) = P^I \to a_0$$

where P^I is the element $P^{(i_1,i_2,\ldots i_{n-1})}$ of the Milnor basis for A and a_0 gnerates $\Sigma^{d(I)}Z/p$. Since ξ^I generates both Q_0 and Q_1 homology this is an E^1 module map. If $i_n\neq 0$ then $L(\nu_p(i_n!))$ is generated by a_i for $0\leq i\leq\nu_p(i_n!)$. If $i_n = i_{n,0}+i_{n,1}\cdot p+\ldots+i_{n,s}\cdot p^s$ is the p-ary expansion of i_n then fix i between 1 and $\nu_p(i_n!)$ and choose j between 1 and s such that

$$\gamma_{j-1}(i_n)<i\leq\gamma_j(i_n).$$

Set

$I(i)=(i_1,\ldots i_{n-1},i_{n,0},i_{n,1},\ldots i_{n,j-2},\beta_{j-1}(i_n)-p(i-\gamma_{j-1}(i_n)),i-\gamma_{j-1}(i_n))$

$J(i) = I(i)-\Delta_{i+j}$

where Δ_n is the sequence with all zeros except a 1 in the n^{th} slot.

Define h_I by

$$h_I(P^I) = a_0$$

$$h_I(P^{I(i)}) = a_i \qquad 1\le i\le \nu_p(i_n!)$$

$$h_I(P^{J(i)}Q_{i+j}) = Q_0a_i \qquad 1\le i\le \nu_p(i_n!)$$

$$h_I(x) = 0 \qquad \text{for all other basis elements in } A/\bar{E}^n\cdot A.$$

Notice that this means $h_I(P^{I(\nu_p(i_n!))}) = a_{\nu_p(i_n!)}$ where

$I(\nu_p(i_n!)) = (i_1,\ldots,i_{n-1},i_{n,0},\ldots,i_{n,s})$. Since for each $i_{n,j}<p$,

$P^{I(\nu_p(i_n!))}$ generates Q_1 homology as does $a_{\nu_p(i_n!)}$, this map is

surjective in Q_0 and Q_1 homology.

Now

$$h:H^*BP\langle n\rangle \longrightarrow A/\bar{E}^n\cdot A + \oplus_{I\in K_n}\Sigma^{d(I)}L(\nu_p(i_n!))$$

is an isomorphism in Q_0 and Q_1 homology. Hence by theorem 2.1 of [AM], the kernel of h is free over E^1 so it must be $V\otimes E^1$ for some graded vector space V of finite type. Since h is surjective we have proved,

<u>Theorem 3.1</u>

$$H^*BP\langle n\rangle\cong\oplus_{I\in K_n}\Sigma^{d(I)}L(\nu_p(i_n!))\oplus V\otimes E[Q_0,Q_1]$$

as modules over $E[Q_0,Q_1]$.

We proceed to calculate the Ext groups for the spectral sequence we have in mind. Table 3.9 of [AP] or direct calculation tells us that

$$\mathrm{Ext}^{**}_{E^1}(F_p,F_p) \cong F_p[q_0,q_1]$$

where q_0 generates $\mathrm{Ext}^{1,1}_{E^1}(F_p,F_p)$ and q_1 generates $\mathrm{Ext}^{1,q+1}_{E^1}(F_p,F_p)$.

For any nonnegative integer m, let $M(m)$ be the graded $F_p[q_0,q_1]$ module given by

generators: a_i, $0 \le i \le m$ $\quad |a_i| = (0,qi)$

relations: $q_1 a_i = q_0 a_{i+1}$ $\quad 0 \le i < m$.

If $m<0$ then $M(m)$ is given by

generators: a, b_i $\quad m \le i \le -1$ $\quad |b_i| = (0,qi-1)$ $\quad |a| = (-m,-m)$

relations: $q_1 b_i = q_0 b_{i+1}$ $\quad m \le i < -1$

$$q_0 b_m = 0 = q_1 b_{-1}.$$

<u>Theorem 3.2</u> ([§3;AP])

For any integer n

$$\mathrm{Ext}^{**}_{E^1}(L(n),F_p) \cong M(n)$$

$$\mathrm{Ext}^{**}_{E^1}(F_p,L(n)) \cong M(-n)$$

as modules over $F_p[q_0,q_1]$.

Now for m a positive integer define modules $N(m)$ over $Z^\wedge_p[v] \cong BP\langle 1\rangle_*$ by

generators: a_i, $0 \le i \le m$ $\quad |a_i| = qi$

relations: $va_i = pa_{i+1}$ $\quad 0 \le i < m$.

If m is a negative integer define N(m) by

generators: $b_i, a \quad m \le i \le -1 \quad |b_i| = qi-1 \quad |a| = 0$

relations: $vb_i = pb_{i+1} \qquad m \le i < -1$

$pb_m = 0 = vb_{-1}.$

N(0) is just $Z_p^\wedge[v]$.

Theorem 3.3

$$BP\langle 1\rangle_* BP\langle n\rangle = \oplus_{I\in K_n} \Sigma^{d(I)} N(\nu_p(i_n!)) \otimes V$$

as a module over $BP\langle 1\rangle_* = Z_p^\wedge[v]$, where V is the F_p vector space of 3.1

Proof Combining theorems 3.1 and 3.2 gives

$$\mathrm{Ext}^{**}_{E^1}(H^*BP\langle n\rangle, F_p) = \oplus_{I\in K_n} \Sigma^{d(I)} M(\nu_p(i_n!)) \otimes V.$$

Since all the elements in positive filtration are torsion free and lie in even stems, all differentials must be zero and the spectral sequence collapses to the E_2 term. In order to solve the extension problem and demonstrate the module structure over $BP\langle 1\rangle_*$ we are going to use the pairing of Adams spectral sequences

$$\begin{array}{ccc} \hat{F}^{s,t} \otimes F^{s',t'} & \to & F^{s+s',t+t'} \\ \downarrow & & \downarrow \\ \hat{E}_\infty^{s,t} \otimes E_\infty^{s',t'} & \to & E_\infty^{s+s',t+t'} \end{array}$$

where $\hat{F}$ and $\hat{E}_\infty$ are the filtration and E_∞ terms of the Adams spectral sequence for [S,BP<1>] and F and E_∞ are the filtration and E_∞ terms of the Adams spectral sequence for $[S, BP\langle 1\rangle \wedge BP\langle n\rangle]$.

As mentioned earlier $\hat{E}^{s,t}$ is the ring $F_p[q_0, q_1]$ and we already know the product structure for the bottom line. Choose an element called p in $\hat{F}^{1,1}$ which maps to q_0 and an element v in $\hat{F}^{1,q+1}$ which maps to q_1. Then elments in $F^{s,t}$ with $s\geq 2$ are in the ideal $(p,v)\cdot F^{s',t}$ with $s'\geq 1$. To choose generators for $BP\langle 1\rangle_* BP\langle n\rangle$ satisfying the relations of $N(\nu_p(i_n!))$ work in order of increasing degree. Choose a generator a_i in $F^{0,qi}$ which maps to the correct element b_i in $M(\nu_p(i_n!))$. Then $pa_i - va_{i-1}$ maps to $q_0a_i - q_1a_{i-1} = 0$ in the associated graded hence it lifts to an element c of $F^{2,*}$. This element is in the ideal $(p,v)\cdot F^{s,t}$ for $s\geq 1$ so a_i and a_{i-1} can be adjusted by an element of positive filtration to ensure that $pa_i - va_{i-1} = 0$. This may require adjusting a_{i-2} and so on but the process stops because $M(\nu_p(i_n!))$ has only finitely many generators. Q.E.D.

CHAPTER 4. THE BP<1> COHOMOLOGY OF BP<n>

In this chapter we compute $BP\langle 1\rangle^* BP\langle n\rangle$. This time in order to see that the Adams spectral sequence collapses we need to have a look at the universal coefficient spectral sequence as well. These two spectral sequences give different filtrations to elements in $BP\langle 1\rangle^* BP\langle n\rangle$ which will allow us to conclude that both collapse to their respective E_2 terms.

<u>Proposition 4.1</u>

In the Adams spectral sequence for computing $[BP\langle n\rangle, BP\langle 1\rangle]$,

$$E_2^{s,t} = \oplus_{I\in K_n} \Sigma^{-d(I)} M(-\nu_p(i_n!)) \oplus \Sigma^{-q-2} V^*$$

as a bigraded module over $F_p[q_0, q_1]$ where V^* is the graded dual of V.

<u>Proof</u> Just combine theorems 3.1 and 3.2 Q.E.D.

For any module N over a ring R write T(N) for the torsion submodule

$$T(N) = \{n\in N \mid rn = 0 \text{ for some } 0 \neq r \in R\}.$$

Then N/T(N) is torsion free. One notices that $\Sigma^{d(I)} T(M(-\nu_p(i_n!)))$ is concentrated in odd stems and $\Sigma^{d(I)} M(-\nu_p(i_n!))/T(M(-\nu_p(i_n!)))$ is concentrated in even stems. As usual V^* is concentrated in filtration zero. Thus the only possible nontrivial differentials in this spectral sequence will have $T(\oplus_{I\in K_n} \Sigma^{d(I)} M(-\nu_p(i_n!)))$ as their target. We turn to the universal coefficient spectral sequence to see that this can not happen.[1]

The spectral sequence of §6 of [R3] specializes in our case to give

$$\mathrm{Ext}^{s,t}_{BP\langle 1\rangle_*}(BP\langle 1\rangle_* BP\langle n\rangle, BP\langle 1\rangle_*) \Longrightarrow [BP\langle n\rangle, BP\langle 1\rangle]_{t-s}$$

[1] Actually, although bu is an A_∞ ring spectrum, it is not known whether BP<1> is. Thus, what the arguments of the next two sections prove is that the Adams spectral sequences for [BP<n>, bu] and $[BP\langle 0\rangle^K, BP\langle n\rangle \wedge bu]$ collapse. To recover $BP\langle 1\rangle^* BP\langle n\rangle$ use $bu \simeq \bigvee_{i=0}^{p-2} \Sigma^{2i} BP\langle 1\rangle$.

Proposition 4.2

$$\mathrm{Ext}^{0,*}_{BP\langle 1\rangle_*}(BP\langle 1\rangle_* BP\langle n\rangle, BP\langle 1\rangle_*) \simeq \bigoplus_{I\in K_n} \Sigma^{-d(I)} N(-\nu_p(i_n!))/T(N(-\nu_p(i_n!)))$$

$$\mathrm{Ext}^{1,*}_{BP\langle 1\rangle_*}(BP\langle 1\rangle_* BP\langle n\rangle, BP\langle 1\rangle_*) \simeq \Sigma\bigoplus_{I\in K_n} \Sigma^{-d(I)} T(N(-\nu_p(i_n!)))$$

$$\mathrm{Ext}^{2,*}_{BP\langle 1\rangle_*}(BP\langle 1\rangle_* BP\langle n\rangle, BP\langle 1\rangle_*) \simeq \Sigma^{-q} V*$$

$$\mathrm{Ext}^{s,*}_{BP\langle 1\rangle_*}(BP\langle 1\rangle_* BP\langle n\rangle, BP\langle 1\rangle_*) \simeq 0 \qquad \text{for } s\geq 3$$

Proof From theorem 3.3 it suffices to show

Lemma 4.3

For n a positive integer

$$\mathrm{Ext}^{s,*}_{Z^\wedge_p[v]}(N(n), Z^\wedge_p[v]) \simeq \begin{cases} Z^\wedge_p[v] & \text{if } s=0 \\ \Sigma T(N(-n)) & \text{if } s=1 \\ 0 & \text{if } s\geq 2 \end{cases}$$

$$\mathrm{Ext}^{s,*}_{Z^\wedge_p[v]}(Z/p, Z^\wedge_p[v]) \simeq \begin{cases} \Sigma^{-q} Z/p & \text{if } s=2 \\ 0 & \text{if } s\neq 2 \end{cases}$$

Proof A free resolution of N(n) is given by

$$0 \longleftarrow N(n) \xleftarrow{\varepsilon} \bigoplus_{i=0}^{n} \Sigma^{qi} Z^\wedge_p[v]\cdot e_i \xleftarrow{d_0} \bigoplus_{i=0}^{n-1} \Sigma^{q(i+1)} Z^\wedge_p[v]\cdot f_i \longleftarrow 0$$

where the $Z^\wedge_p[v]$ linear maps are defined by

$$\varepsilon(e_i) = a_i$$

$$d_0(f_i) = ve_i - pe_{i+1}.$$

A free resolution of Z/p is given by

$$0 \leftarrow Z/p \xleftarrow{\varepsilon} Z^\wedge_p[v]\cdot e \xleftarrow{d_0} Z^\wedge_p[v]\cdot f_1 \oplus \Sigma^q Z^\wedge_p[v]\cdot f_2 \xleftarrow{d_1} \Sigma^q Z^\wedge_p[v]\cdot g \leftarrow 0$$

where the maps are

$$\varepsilon(e) = 1$$

$$d_0(f_1) = pe$$

$d_0(f_2) = ve$

$d_1(g) = vf_1 - pe_2$.

Applying $\mathrm{Hom}_{\hat{Z}_p[v]}(\ \ ,\hat{Z}_p[v])$ to these resolutions and taking homology gives the desired results. Q.E.D.

Theorem 4.4

$$[BP\langle n\rangle, BP\langle 1\rangle]_* = \oplus_{I\in K_n} \Sigma^{-d(I)} N(-\nu_p(i_n!)) \otimes \Sigma^{-q-2} V_*$$

as a module over $\hat{Z}_p[v]$ because both the Adams spectral sequence and the universal coefficient spectral sequence collapse to their E_2 terms.

Proof In the universal coefficient spectral sequence, the $E_2^{1,*}$ term automatically survives to $E_\infty^{s,*}$ because $d_r: E_r^{s,*} \to E_r^{s+r,*}$ and $E_2^{s,*}$ is zero for $s \geq 3$. Hence the torsion modules $\oplus_{I\in K_n} \Sigma^{-d(I)} T(N(-\nu_p(i_n!)))$ appear in $E_\infty^{1,*}$. Since they are concentrated in odd stems and all the torsion free groups are in even stems, for each summand there is a torsion group at least as large (possibly larger because of the possibility of extensions with the Z/p's in filtration two) to $[BP\langle n\rangle, BP\langle 1\rangle]$. The only way the Adams spectral sequence can account for this is if all the $T(N(-\nu_p(i_n!)))$ survive to the E_∞ term, but recall that they were the only possible targets for differentials so the Adams spectral sequence collapses. The extensions corresponding to those envisaged in the universal coefficient spectral sequence can not occur in the Adams spectral sequence. Q.E.D.

A review of the proof will indicate that what we have really proved is:

<u>Theorem 4.5</u>

If X is a connective spectrum of finite type with

$$H^*X \cong \oplus_{r\in R}\, \Sigma^{i_r} L(j_r) \oplus V\otimes E^1$$

as a module over E^1 with $i_r \equiv i_s \bmod 2$ for all r and s and j_r nonnegative for all r then

(a) the Adams spectral sequence for [S,BP<1>∧X] collapses to give

$$BP\langle 1\rangle_* X \cong \oplus_{r\in R}\, \Sigma^{i_r} N(j_r) \oplus V$$

as a module over $BP\langle 1\rangle_*$ and

(b) the Adams spectral sequence for [X,BP<1>] collapses to give

$$[X, BP\langle 1\rangle]_* \cong \oplus_{r\in R}\, \Sigma^{-i_r} N(-j_r) \oplus \Sigma^{-q-2} V^*.$$

CHAPTER 5 A SPLITTING OF $BP\langle 1\rangle \wedge BP\langle n\rangle$

Spectra $BP\langle 0\rangle^k$ with $H^*BP\langle 0\rangle^k \cong A/A\cdot\{\beta, \chi P^i \mid i>k\}$ were constructed (with different names) by Mahowald [Mah1] for $p=2$ and by Kane [K] for odd primes p. In section 11 of [K], Kane proves that $H^*\Sigma^{qk}BP\langle 0\rangle^k$ is isomorphic as an E^1 module to monomials of weight pk in $F_p[\xi_1, \xi_2, \ldots] \otimes E[\tau_2, \tau_3, \ldots]$, which implies that $H^*\Sigma^{qk}BP\langle 0\rangle^k$ is isomorphic as an E^1 module to $\Sigma^{qk} L(\nu_p((pk)!)) \otimes V \otimes E^1$ for some graded F_p vector space V. Thus there is an isomorphism in cohomology

$$H^*(BP\langle 1\rangle \wedge BP\langle n\rangle \vee HV) \cong \oplus_{I\in K_n} H^*(BP\langle 1\rangle \wedge \Sigma^{d(I)} BP\langle 0\rangle^{[i_n/p]}) \oplus H^*HV'$$

for some graded F_p vector spaces V and V'. The goal of this chapter is to prove that this isomorphism is realized by a homotopy equivalence of spectra.

<u>Theorem 5.1</u>

$$BP\langle 1\rangle \wedge BP\langle n\rangle \vee HV \cong \vee_{I\in K_n} BP\langle 1\rangle \wedge \Sigma^{d(I)} BP\langle 0\rangle^{[i_n/p]} \vee HV'$$

In order to construct this splitting we want to construct maps in $[BP\langle 0\rangle^k, BP\langle 1\rangle \wedge BP\langle n\rangle]$ which will realize this isomorphism in cohomology after they have been smashed with $BP\langle 1\rangle$. In order to do this it suffices to show that for arbitrary $k\geq 0$ the Adams spectral sequence for $[BP\langle 0\rangle^k, BP\langle 1\rangle \wedge BP\langle n\rangle]$ collapses. As in chapter four we contrast the associated graded which the Adams spectral sequence gives for this group with the one given by the universal coefficient spectral sequence

$$\mathrm{Ext}^{s,t}_{BP\langle 1\rangle_*}(BP\langle 1\rangle_* BP\langle 0\rangle^k, BP\langle 1\rangle_* BP\langle n\rangle) \Longrightarrow [BP\langle 0\rangle^k, BP\langle 1\rangle \wedge BP\langle n\rangle]_{t-s}$$

a special case of the universal coefficient spectral sequence found in §6 of [R3].

First let us record the E_2 term of the Adams spectral sequence for $[BP\langle 0\rangle^k, BP\langle 1\rangle \wedge BP\langle n\rangle]$.

<u>Proposition 5.2</u>

$$\mathrm{Ext}_{E^1}(H^*BP\langle n\rangle, H^*BP\langle 0\rangle^k)$$

$$\cong \mathrm{Ext}_{E^1}(\oplus_{I\in K_n} \Sigma^{d(I)}L(\nu_p(i_n!)) \oplus V\otimes E^1, \Sigma^{qk}L(\nu_p((pk)!)) \oplus V'\otimes E^1)$$

$$\cong \oplus_{I\in K_n} \Sigma^{d(I)}M(\nu_p(i_n!)-\nu_p((pk)!)) \oplus \Sigma^{-1}(\oplus_{j=0}^{\nu_p(i_n!)} \oplus_{i=1}^{k} \Sigma^{q(j-i)}Z/p)$$

$$\oplus V\otimes(\Sigma^{qk}L(\nu_p((pk)!))^* \oplus V\otimes(V'\otimes E^1)^*$$

$$\oplus\Sigma^{-q-2}\oplus_{I\in K_n} \Sigma^{d(I)}L(\nu_p(i_n!))\otimes V'^*$$

<u>Proof</u> As in chapter three, use section 3 of [AP] to determine Ext^s for $s\geq 1$ and compute $\mathrm{Ext}^0 = \mathrm{Hom}$ directly. Q.E.D.

To apply the universal coefficient spectral sequence we need to know $BP\langle 1\rangle_* BP\langle n\rangle$ which is given in chapter three and $BP\langle 1\rangle_* BP\langle 0\rangle^k$.

<u>Proposition 5.3</u>

$$BP\langle 1\rangle_* BP\langle 0\rangle^k \cong N(\nu_p((pk)!))\oplus V'$$

where V' is as in 5.2.

<u>Proof</u> Exactly the same as the computation of $BP\langle 1\rangle_* BP\langle n\rangle$ in theorem 3.3. Q.E.D.

The last ingredient needed to use the universal coefficient spectral sequence is the Ext groups which will appear.

<u>Proposition 5.4</u>

(a)

$$\mathrm{Ext}^{s,*}_{BP\langle 1\rangle_*}(F_p,F_p) = \begin{cases} Z/p & s=0 \\ Z/p \oplus \Sigma^{-q}Z/p & s=1 \\ \Sigma^{-q}Z/p & s=2 \\ 0 & s\geq 3 \end{cases}$$

so it is isomorphic to E^{1*} as a graded F_p vector space.

(b)

$$\mathrm{Ext}^{s,*}_{BP\langle 1\rangle_*}(F_p,N(m)) = \begin{cases} 0 & s=0 \\ \oplus_{i=0}^{m-1} \Sigma^{qi}Z/p & s=1 \\ \Sigma^{-q}(\oplus_{i=0}^{m} \Sigma^{qi}Z/p & s=2 \\ 0 & s\geq 3 \end{cases}$$

so it is isomorphic to $\Sigma^{-q-2}L(m)$ as a graded F_p vector space.

(c)

$$\mathrm{Ext}^{s,*}_{BP\langle 1\rangle_*}(N(m),F_p) = \begin{cases} \oplus_{i=0}^{m} \Sigma^{-qi}Z/p & s=0 \\ \oplus_{i=1}^{m} \Sigma^{-qi}Z/p & s=1 \\ 0 & s\geq 2 \end{cases}$$

so it is isomorphic to $L(m)^*$ as a graded F_p vector space.

(d)

$$\mathrm{Ext}^{s,*}_{BP\langle 1\rangle_*}(N(m),N(n)) = \begin{cases} N(n-m)/T(N(n-m)) & s=0 \\ \Sigma T(N(n-m)) \oplus \oplus_{j=0}^{n} \oplus_{i=1}^{m} \Sigma^{q(j-i)}Z/p & s=1 \\ 0 & s\geq 2 \end{cases}$$

<u>Proof</u> (a) $0 \leftarrow F_p \leftarrow Z_p^{\hat{}}[v] \leftarrow Z_p^{\hat{}}[v] \oplus \Sigma^q Z_p^{\hat{}}[v] \leftarrow \Sigma^q Z_p^{\hat{}}[v] \leftarrow 0$

$$1 \leftarrow 1 \qquad\qquad (-v,p) \leftarrow 1$$

$$p \leftarrow (1,0)$$

$$v \leftarrow (0,1)$$

is a free resolution of F_p. Applying $\mathrm{Hom}_{BP\langle 1\rangle_*}(\quad ,F_p)$ gives

$$F_p \xrightarrow{0} F_p \oplus \Sigma^{-q}F_p \xrightarrow{0} \Sigma^{-q}F_p$$

so that taking the s^{th} homology group gives the desired result.

(b) Applying $\mathrm{Hom}_{BP\langle 1\rangle_*}(\quad, N(m))$ gives

$$N(m) \to N(m) \oplus \Sigma^{-q}N(m) \to \Sigma^{-q}N(m)$$

$$x \to (px, vx)$$

$$(x,y) \to py - vx$$

The s^{th} homology group of this cochain complex gives the desired Ext groups so that $\mathrm{Ext}^{0,*}_{BP\langle 1\rangle_*}(F_p, N(m)) \approx 0$, $\mathrm{Ext}^{1,*}_{BP\langle 1\rangle_*}(F_p, N(m)) \approx \oplus_{i=0}^{m-1} \Sigma^{qi} Z/p$

generated by pairs (a_i, a_{i+1}) with $0 \le i \le m-1$ and $\mathrm{Ext}^{2,*}_{BP\langle 1\rangle_*}(F_p, N(m))$

$\approx \Sigma^{-q}(\oplus_{i=0}^{m} \Sigma^{qi} Z/p)$ generated by a_i for $0 \le i \le m$.

(c) Applying $\mathrm{Hom}_{BP\langle 1\rangle_*}(\quad, F_p)$ to the free resolution of $N(m)$ found in the proof of 4.3 gives

$$\oplus_{i=0}^{m} \Sigma^{qi} Z/p \xrightarrow{0} \oplus_{i=1}^{m} \Sigma^{qi} Z/p$$

Taking the s^{th} homology group gives the desired result.

(d) The proof is by induction on m. For $m=0$,

$$\mathrm{Ext}^{0,*}_{BP\langle 1\rangle_*}(BP\langle 1\rangle_*, N(n)) \approx \mathrm{Hom}_{BP\langle 1\rangle_*}(BP\langle 1\rangle_*, N(n)) \approx N(n)$$

and

$$\mathrm{Ext}^{s,*}_{BP\langle 1\rangle_*}(BP\langle 1\rangle_*, N(n)) \approx 0 \qquad s \ge 1$$

as stated.

The short exact sequence

$$0 \to \Sigma^{q}N(m) \to N(m+1) \to BP\langle 1\rangle_*/(v) \to 0$$

$$a_i \to a_{i+1}$$

$$a_0 \to 1$$

induces

$$0 \to \mathrm{Ext}^0(Z_p^{\wedge}, N(n)) \to \mathrm{Ext}^0(N(m+1), N(n)) \to \mathrm{Ext}^0(\Sigma^q N(m), N(n))$$

$$\xrightarrow{\delta} \mathrm{Ext}^1(Z_p^{\wedge}, N(n)) \to \mathrm{Ext}^1(N(m+1), N(n)) \to \mathrm{Ext}^1(\Sigma^q N(m), N(n)) \to 0.$$

Calculation with a free resolution of $Z_p^\wedge = BP\langle 1\rangle_*/(v)$ indicates that $Ext^0(Z_p^\wedge,N(m)) = 0$ and $Ext^1(Z_p^\wedge,N(n)) = N(n)/(v)\cdot N(n) = Z_p^\wedge \oplus (\oplus_{i=1}^n \Sigma^{qi} Z/p)$. The connecting homomorphism δ, is just the Yoneda product with the element of $Ext^1(Z_p^\wedge,N(m)) = N(m)/(v)\cdot N(m)$ represented by the short exact sequence which can be calculated to be pa_0. From this we can see that

$$Ext^0(N(m+1),N(n)) = Ext^0(\Sigma^q N(m),N(m))/\mathrm{coim}\ \delta$$

which by induction gives the desired result for $Ext^0(N(m+1),N(n))$. Similarly

$$Ext^1(N(m+1),N(n)) = Ext^1(\Sigma^q N(m),N(n))\oplus \mathrm{coker}\ \delta$$

gives the desired result for $Ext^1(N(m+1),N(n))$. Q.E.D.

Before launching into the proof of 5.1 it is enlightening to consider the following chart which summarizes the different associated gradeds for $[BP\langle 0\rangle^k, BP\langle 1\rangle \wedge BP\langle n\rangle]$ provided by the two different spectral sequences.

Summand in $H^*BP\langle 0\rangle^k$	Summand in $H^*BP\langle n\rangle$	Contribution to E^2 of A.S.S.	Contribution to E^2 of U.C.S.S.
$L(n)$	$L(m)$	$\Sigma^{-1}\oplus_{j=0}^m \oplus_{i=1}^n \Sigma^{q(j-i)} Z/p$ in $s=0$ $\oplus$ $M(m-n)$	$N(m-n)/T(N(m-n))$ in $s=0$ $\oplus$ $\oplus_{j=0}^m \oplus_{i=1}^n \Sigma^{q(j-i)} Z/p$ $\oplus$ $T(N(m-n))$ in $s=1$
$L(n)$	E^1	$L(n)^*$ in $s=0$	$L(n)^*$ split between $s=0$ and $s=1$
E^1	$L(m)$	$\Sigma^{-q-2}L(m)$ in $s=0$	$\Sigma^{-q-2}L(m)$ split between $s=1$ and $s=2$
E^1	E^1	E^{1*} in $s=0$	E^{1*} split between $s=0,1$ and 2

Proof of 5.1 Because $BP\langle 1\rangle_*$ has global dimension 2, everything in $E_2^{1,*}$ of the universal coefficient spectral sequence automatically survives to the E_∞ term and contributes to $[BP\langle 0\rangle^k, BP\langle 1\rangle \wedge BP\langle n\rangle]$. Since all the torsion free groups are in even stems while the torsion in $E_2^{1,*}$ is in odd stems, we know that each summand in $\oplus_{I\in K_n} \Sigma^{-d(I)} T(M(\nu_p(i_n!)-\nu_p((pk)!))$ contributes a torsion group as least as large (may be larger due to the possibility of extensions with some Z/p's in $E_\infty^{2,*}$) to $[BP\langle 0\rangle^k, BP\langle 1\rangle \wedge BP\langle n\rangle]$. The only way the Adams spectral sequence can account for this is if all the $T(M(\nu_p(i_n!)-\nu_p((pk)!))$ survive to the E_∞ term. But as before they are the only possible targets for differentials since all other torsion is in filtration zero and all the torsion free towers are in even stems. This time there is no possibility of extensions with the corresponding Z/p's. Q.E.D.

CHAPTER 6. GENERALIZED BROWN-GITLER SPECTRA OVER BP<2>

In their preprint [GJM] Goerss, Jones and Mahowald proved

Theorem 6.1

For each $k \geq 0$ there exists a p-complete spectrum $BP\langle 1\rangle^k$ and a map $w: BP\langle 1\rangle^k \to BP\langle 1\rangle$ so that

(a) $H^*BP\langle 1\rangle^k \cong A/A\{\beta, \beta P^1, \chi P^i \mid i>k\}$

(b) For $m \leq 2p(k+1)-1$, and Z a CW complex, $w_*: BP\langle 1\rangle^k_m Z \to BP\langle 1\rangle_m Z$ is onto $\cap_{i>k} \ker \chi\bar{P}^j$ where $\chi\bar{P}^j: BP\langle 1\rangle \to \Sigma^{qj} H\hat{Z}_p$ is defined to make

$$\begin{array}{ccc} BP\langle 1\rangle & \xrightarrow{\chi\bar{P}^j} & \Sigma^{qj} H\hat{Z}_p \\ \downarrow & & \downarrow \\ H & \xrightarrow{\chi P^j} & \Sigma^{qj} H \end{array}$$

commute.

They call this the generalized Brown-Gitler spectrum over BP<1>. Our object in this chapter is to prove the existence of generalized Brown-Gitler spectra over BP<2>. First we need

Proposition 6.2

There is a map $S^j: BP\langle 2\rangle \to \Sigma^{qj} BP\langle 1\rangle$ such that

$$\begin{array}{ccc} BP\langle 2\rangle & \xrightarrow{S^j} & \Sigma^{qj} BP\langle 1\rangle \\ \downarrow & & \downarrow \\ H & \xrightarrow{\chi P^j} & \Sigma^{qj} H \end{array}$$

commutes where the vertical maps are the usual reduction maps.

Proof This follows immediately from the collapsing of the Adams spectral sequence for $BP\langle 1\rangle^{*}BP\langle 2\rangle$ proved in theorem 4.1. Q.E.D.

Theorem 6.3

For each $k \geq 0$ there is a p-complete spectrum $BP\langle 2\rangle^{k}$ and a map $w : BP\langle 2\rangle^{k} \to BP\langle 2\rangle$ such that

(a) $H^{*}BP\langle 2\rangle^{k} \to A/A\{\beta, \beta P^{1}, \beta P^{p+1}, \chi P^{j} \mid j>k\}$

(b) For $m \leq 2p(k+1)-1$

$$\{\mathrm{Im}\ w_{*} : BP\langle 2\rangle^{k}_{m}Z \to BP\langle 2\rangle_{m}Z\} \subset \cap_{j>k} \ker S^{j}$$

If $H\hat{Z}_{p*}Z$ is an F_{p} vector space then Im w_{*} is onto $\cap_{j>k} \ker S^{j}$.

Notice that (b) of 6.1 is really a consequence of the tower which has $BP\langle 1\rangle^{k}$ as its inverse limit. A similar statement, stronger than 6.3(b) can be extracted from the tower used to construct $BP\langle 2\rangle^{k}$ but its statement is best left to the reader.

In most of what follows we are simply adjusting the results of [GJM] to the next case up from theirs. Consequently when the proofs are no different, results will just be stated in order to fix notation and direction. For some proofs requiring minor modification I have given the proof with modification. The only results in this chapter which require substantial new ideas are; the proof of the collapsing of the Adams spectral sequence for $BP\langle 1\rangle_{*}(\wedge_{t} BZ/p)$ which was shown to me by P. Goerss, checking that $d_{0} \ast h_{I} = 0$ for the adaptive complexes in section 6.2, one of the early steps in constructing the tower which has $BP\langle 2\rangle^{k}$ as its inverse limit in section 6.3, and of course the construction of the maps $S^{j} : BP\langle 2\rangle \to \Sigma^{qj} BP\langle 1\rangle$.

In this chapter all spectra are completed at p.

6.1 Acyclic Resolutions

Let

$$M_{-1}(k) = A/A\{\chi(\beta^{\varepsilon}P^{i}) \mid i>k, \varepsilon=0,1\}$$

$$M_0(k) = A/A\{\beta, \chi P^{i} \mid i>k\}$$

$$M_n(k) = A/A\{\beta, \beta P^{1}, \ldots, \beta P^{p^{n-1}+p^{n-2}+\cdots+p+1}, \chi P^{i} \mid i>k\} \text{ for } n\geq 1.$$

Notice that $M_2(k)$ will be $H^*BP\langle 2\rangle^k$. We present acyclic resolutions of these modules for $n\leq 2$. Except in the case of $M_{-1}(k)$ they are not free resolutions. To construct them we need the Λ algebra.

For p an odd prime let $\bar{\Lambda}$ be the associative graded algebra over F_p given by

generators: λ_{i-1}, $i\geq 0$, $|\lambda_{i-1}| = qi-1$

μ_{i-1}, $i\geq 0$, $|\mu_{i-1}| = qi$

relations: $\lambda_r\lambda_s=\Sigma_i a(i,r,s)\lambda_{r+s+1-i}\lambda_{i-1}$

$\mu_r\lambda_s=\Sigma_i b(i,r,s)\mu_{r+s+1-i}\lambda_{i-1}$

$\lambda_r\mu_s=\Sigma_i a(i,r,s)\lambda_{r+s+1-i}\mu_{i-1}+c(i,r,s)\mu_{r+s+1-i}\lambda_{i-1}$

$\mu_r\mu_s=\Sigma_i b(i,r,s)\mu_{r+s+1-i}\mu_{i-1}$

where

$$a(i,r,s) = (-1)^{i+r}\binom{(p-1)(s+1-i)-1}{i-p(r+1)}$$

$$b(i,r,s) = (-1)^{i+r+1}\binom{(p-1)(s+1-i)-1}{i-p(r+1)-1}$$

$$c(i,r,s) = (-1)^{i+r}\binom{(p-1)(s+1-i)}{i-p(r+1)}$$

Let Λ be $\bar{\Lambda}$ modulo the left ideal generated by λ_{-1}, i.e. $\Lambda=\bar{\Lambda}/\bar{\Lambda}\{\lambda_{-1}\}$. Λ is a differential graded algebra with differential given by left multiplication by λ_{-1}.

For p=2 let $\bar{\Lambda}$ be the graded associative F_2 algebra given by

generators: λ_i, $i\ge -1$, $|\lambda_i|=i$

relations: $\lambda_r\lambda_s = \Sigma_i \binom{i-1}{2i-(s-2r)} \lambda_{r+i}\lambda_{s-i}$.

Again Λ is $\bar{\Lambda}$ modulo the left ideal generated by λ_{-1}.

If p is odd let $v_i=\lambda_i$ or μ_i. If p=2 then $v_i=\lambda_i$. If $I=(i_1,\ldots,i_q)$ is a q-tuple of integers, $i_j\ge -1$ write v_I for $v_{i_1}\cdot\ldots\cdot v_{i_q}$. This is a monomial of length q. Such a monomial is admissible if

$p(i_j+1) \ge i_{j+1}+2$ when $v_{i_j}=\lambda_{i_j}$

$p(i_j+1) \ge i_{j+1}+1$ when $v_{i_j}=\mu_{i_j}$.

This is valid at any prime. If p is odd set

$$\Lambda_k = \Lambda\{\mu_{-1},\lambda_0,\mu_0,\ldots,\lambda_{k-1},\mu_{k-1}\}$$

and if p=2

$$\Lambda_k = \Lambda\{\lambda_0,\ldots,\lambda_{2k}\}.$$

Lemma 6.4 ([C],[BG],[BCKQRS])

(a) For all primes, a basis for Λ as an F_p vector space is given by the set of admissible monomials.

(b) Λ_k is closed under the differential and has an F_p basis of admissible monomials v_I with $i_q\le k-1$ if p is odd or $i_q\le 2k$ if p=2.

This lemma implies that Λ/Λ_k inherits a differential from Λ given by the left action of λ_{-1} and that Λ/Λ_k has an F_p basis of admissible monomials v_I with $i_q \geq k$ if p is odd or $i_q > 2k$ if $p=2$.

Let $\Lambda_{q,k}$ be the set of admissible monomials of length q in Λ/Λ_k. Define

$$d_q^*: A\otimes\Lambda_{q+1,k}^* \to A\otimes\Lambda_{q,k}^*$$

to be the A linear map given by

$$d_q^* v_I^* = \Sigma_J \langle v_I^*, \lambda_j v_J \rangle \chi P^{j+1} v_J^* + \langle v_I^*, \lambda_j v_J \rangle \chi \beta P^{j+1} v_J^* \qquad (I)$$

for $p>2$ and

$$d_q^* \lambda_I^* = \Sigma_J \langle \lambda_I^*, \lambda_j \lambda_J \rangle \chi Sq^{j+1} \lambda_J^* \qquad (I)$$

for $p=2$. The sums are over $j \geq -1$ and all basis elements v_J of $\Lambda_{q,k}$.

Theorem 6.5 ([C],[BG])

The following is a long exact sequence.

$$\cdots \to A\otimes\Lambda_{q+1,k}^* \xrightarrow{d_q^*} A\otimes\Lambda_{q,k}^* \to \cdots \to A\otimes\Lambda_{1,k}^* \xrightarrow{d_0^*} A \longrightarrow M_{-1}(k) \to 0$$

Now fix $k \geq 0$ and let K_q be an Eilenberg-MacLane spectrum such that $\pi_* K_q = \Lambda_{q,k}$ and let

$$d_q: K_q \to \Sigma K_{q+1}$$

be a map so that in cohomology d_q^* is as in theorem 6.5.

Theorem 6.6 ([GJM])

There exist spectra Y_q and maps $e_q: Y_q \to \Sigma K_{q+1}$ so that

(a) $Y_0 = K_0 = HZ/p$, $e_0 = d_0$

(b) $K_{q+1} \xrightarrow{i_{q+1}} Y_{q+1} \xrightarrow{p_q} Y_q \xrightarrow{e_q} \Sigma K_{q+1}$ is a cofibration

(c) $e_q i_q = d_q$

(d) The induced map of homology theories

$e_{q*}:(Y_q)_m X \to (K_{q+1})_m X$

is zero for $m \leq 2p(k+1)-1$.

(e) If we write B(k) for the homotopy inverse limit of

$\ldots \to Y_q \to Y_{q-1} \to \ldots \to Y_1 \to Y_0 = HZ/p$

then the B(k) are the Brown-Gitler spectra of [BG].

Let Z be a finite complex and $\hat{a} \in H_m Z$ for some $m \leq 2p(k+1)-1$. Then there is a class $a \in H^0 \Sigma^m DZ$ which is dual to $\hat{a}$ which may be realized as a map

$a:\Sigma^m DZ \to HZ/p = Y_0$.

Lemma 6.7 ([GJM])

Any lifting of a to Y_q lifts to Y_{q+1}.

Define a left ideal Λ^n over Λ by

$\Lambda^n = \Lambda\cdot\{\lambda_{i_0}\cdot\ldots\cdot\lambda_{i_n} \mid i_j \geq 0,\ 0 \leq j \leq n\}$ if $p \neq 2$

$\Lambda^n = \Lambda\cdot\{\lambda_{2i_0+1}\cdot\ldots\cdot\lambda_{2i_n+1} \mid i_j \geq 0,\ 0 \leq j \leq n\}$ if $p=2$

Lemma 6.8

Λ^n is closed under the differential of Λ. If $p>2$ then Λ^n has a basis of admissible monomials $v_{i_1}\cdot\ldots\cdot v_{i_q}$, $q \geq n$, with $v_{i_m} = \lambda_{i_m}$ for $q-n \leq m \leq q$. If $p=2$ a basis is given by admissible monomials $\lambda_{i_1}\cdot\ldots\cdot\lambda_{i_q}$, $q \geq n$, with $i_m \equiv 1 \bmod 2$ for $q-n \leq m \leq q$.

Proof This is an immediate consequence of the relations for the Λ algebra. Q.E.D.

Define

$$\Lambda_k^n = \Lambda^n/\Lambda^n \cap \Lambda_k$$

$\Lambda(q,n)$ = monomials of length q in Λ_k^n if $q>n$

$\Lambda(q,n) = \Lambda(q,q-1)$ if $q \le n$

Let

$$D_q^n = A \otimes \Lambda^*(q,n) \quad \text{if } q>n$$

$$D_n^n = A/A\beta \otimes \Lambda^*(n,n)$$

$$D_{n-k}^n = A/A\{\beta, \beta P^1, \ldots, \beta P^{p^{k-1}+p^{k-2}+\cdots+p+1}\} \otimes \Lambda^*(n-k,n) \quad \text{for } 0<k<n$$

$$D_0^n = A/A\{\beta, \beta P^1, \ldots, \beta P^{p^{n-1}+\cdots+1}\} \quad \text{i.e. } \Lambda(0,0)=Z/p.$$

Define $d_q^*: D_{q+1}^n \to D_q^n$ by the formulae (I).

<u>Theorem 6.9</u>

The following is a long exact sequence of A modules.

$$\cdots \to D_{q+1}^2 \xrightarrow{d_q^*} D_q^2 \to \cdots \to D_1^2 \xrightarrow{d_1^*} D_0^2 \xrightarrow{\varepsilon} M_2(k) \to 0$$

<u>Proof</u> Since $P^1\beta P^n = n\beta P^{n+1} + P^{n+1}\beta$,

$$P^pP^1\beta P^nP^m = -\binom{(p-1)(n+1)}{p}\beta P^{n+p+1}P^m - n\beta P^{n+p}P^1P^m$$
$$+\binom{(p-1)(n+1)-1}{p-1}P^{n+p+1}\beta P^m - mP^{n+p}\beta P^{m+1} - P^{n+p}P^{m+1}$$

we find that P^I with I admissible and $\varepsilon_1=\varepsilon_2=0$ form a basis for $A/\{\beta, P^1\beta, P^pP^1\beta\}\cdot A$. Hence P^I with I admissible and $\varepsilon_1=\varepsilon_2=0$ form a basis for $A/A\{\beta, \beta P^1, \beta P^{p+1}\}$. Now the proofs of theorem 1.7 of [C] and theorem 2.7 of [BG] go through verbatim. Q.E.D.

Let $C_q = A \otimes \Lambda_{q,k}^*$. Then there are maps $\Theta_q^*: C_q \to D_q^2$ so that

$$\begin{tikzcd} C_{q+1} \arrow[r, "d_q^*"] \arrow[d, "\Theta_{q+1}^*"] & C_q \arrow[d, "\Theta_q^*"] \\ D_{q+1}^2 \arrow[r, "d_q^*"] & D_q^2 \end{tikzcd}$$

commutes. If q is greater than 2 then C_q and D_q^2 are both direct sums of copies of A, the collection D_q^2 being a subcollection of C_q and Θ_q^* is the projection. If the K_q are as in theorem 6.5 then $H^*K_q = C_q$. Set $K_0^2 = BP\langle 2\rangle$, $K_1^2 = \vee_{i>k}\Sigma^{qi-1}BP\langle 1\rangle$, and $K_2^2 = \vee_{(i,j)}\Sigma^{q(i+j)-2}H\hat{Z}_p$, where this last sum is over pairs (i,j) such that $\lambda_i\lambda_j \in \Lambda(2,2)$. For $q\geq 3$ set K_q^2 to be the Eilenberg-MacLane spectrum with $\pi_* K_q^2 = \Lambda(q,2)$.

<u>Proposition 6.10</u>

For $q\geq 0$ there exist maps $\Theta_q : K_q^2 \to K_q$ and maps $d_q : K_q^2 \to \Sigma K_{q+1}^2$ so that

(a) $\Theta_q^* : C_q \to D_q^2$ is the quotient and $d_q^* : D_{q+1}^2 \to D_q^2$ is the differential.

(b) The following diagram commutes,

$$\begin{tikzcd} K_q^2 \arrow[r, "d_q"] \arrow[d, "\Theta_q"] & \Sigma K_{q+1}^2 \arrow[d, "\Sigma\Theta_{q+1}"] \\ K_q \arrow[r, "d_q"] & \Sigma K_{q+1} \end{tikzcd}$$

(c) If $q\geq 3$ then there is $s_q : K_q \to K_q^2$ so that $s_q\Theta_q$ is the identity.

<u>Proof</u> The only maps that we need to worry about are

$d_0 : BP\langle 2\rangle \to \vee_{i>k}\Sigma^{qi}BP\langle 1\rangle$

and

$$d_1: \vee_{i>k}\Sigma^{qi}BP\langle 1\rangle \to \vee_{(i,j)}\Sigma^{q(i+j)}HZ_p.$$

Just let d_0 be the product of the S^j in proposition 6.5 noticing that $\Pi_{i>k}\Sigma^{qi}BP\langle 1\rangle \simeq \vee_{i>k}\Sigma^{qi}BP\langle 1\rangle$. Construct d_1 using the appropriate wedges and products of the $\chi\bar{P}^j$ of theorem 2 in [GJM]. Q.E.D

6.2 Adapted Complexes

Let E be a ring spectrum with $H^*E \simeq A/A\cdot\bar{B}$ where $\bar{B}$ is the nonunits of B, a subalgebra of A. Write N(k) for $A/A\{\bar{B},W_k\}$ where $W_k \subset \{\chi P^i \mid i>k\}$. Let $1:E \to H$ be the generator of H^*E over A.

Definition

If Z is a finite CW complex and $h' \in E_m Z$ with $m \le 2p(k+1)-1$ and $h:\Sigma^m DZ \to E$ the dual of h' then we say (Z,h') is adapted to N(k) if

$$A\{\bar{B},W_k\} \longrightarrow A \xrightarrow{h^*\circ 1^*} H^*\Sigma^m DZ$$

is exact. This is an adapted complex of degree m since $h' \in E_m Z$.

The aim of this section is to prove:

Theorem 6.11

For each $k \ge 0$ there is a finite CW complex Z_k and $h_k \in BP\langle 2\rangle_m Z_k$ with $m = 2pk+3$ so that (Z_k, h_k) is adapted to $M_2(k)$. h_k can be chosen so that $d_{0*}h_k = 0$, where d_0 is as in proposition 6.10.

As aids for building adapted complexes we need the following two propositions.

Proposition 6.12 ([GJM])

Any $a \in A$ induces a right dual action $a^*:H_*Z \to H_*Z$. This map is induced by $\chi a:H \to H$. If $G:H_n Z \to H^{-n}DZ$ is the duality isomorphism then

$$\begin{tikzcd}
H_n Z \arrow[r, "a^*"] \arrow[d, "G"] & H_{n-s} Z \arrow[d, "G"] \\
H^{-n} DZ \arrow[r, "\chi a"] & H^{-n+s} DZ
\end{tikzcd}$$

commutes.

Let $\varepsilon : A \to N(k)$ be the quotient map.

Lemma 6.13 ([GJM])

Let $R \subset A$ be a finite set of elements so that $\varepsilon(R)$ is a basis for $N(k)$. Suppose further that for each $P \in R$ there is a finite CW complex Z_P and $h_P \in H_m Z_P$ for some m independent of P. Suppose further that

(a) $\chi P^* h_P \neq 0$ and $m \leq 2p(k+1)-1$

(b) $h_P = 1_* h'_P$ for some h'_P $E_* Z_P$

If $Z = \vee_{P \in R} Z_P$ and $h' = \oplus_P h'_P \in E_m Z$ then (Z,h') is adapted to $N(k)$.

Now our complexes Z_k will be finite subcomplexes of the spaces

$C(s,t) = \Lambda_s CP^\infty \wedge \Lambda_t BZ/p.$

Define a vector space map

$pr : H^*C(s,t) \to H^*C(s,t)/\bar{E}^n \cdot H^*C(s,t) = V.$

$H^{2s+t}C(s,t)$ is generated by a single element c and $H^nC(s,t)=0$ for $n<2s+t$. If $a \in A$ with $v=pr(a \cdot c) \neq 0$ in V then

$a^* pr^* v^* \neq 0.$

Lemma 6.14

Let $I=(i_1,\varepsilon_1,\dots,i_m,\varepsilon_m)$ be admissible with $\varepsilon_1=\varepsilon_2=0$. Set $e=e(I)=2i_1-\Sigma_{j=2}^{m} q i_j - \Sigma_{j=3}^{m}\varepsilon_j$, $t=\Sigma_{i=3}^{m}\varepsilon_i$ and $s=(e-t)/2$. If

$c \in H^{e+3}C(s,t+3)$ is the generator then

$$\mathrm{pr}\,P^I c \neq 0 \text{ in } H^*C(s,t+3)/\bar{E}^n H^*C(s,t+3).$$

Proof Just need to check that $Q_2Q_1Q_0P^I c \neq 0$. This follows from the Adem relations and the fact that in I, $\epsilon_1 = \epsilon_2 = 0$, once one notices that $Q_2Q_1Q_0 = \beta P^{p+1}\beta P^1\beta$ Q.E.D.

Theorem 6.15

For any n, the Adams spectral sequence converging to $BP\langle n\rangle_*C(s,t)$ collapses to its E_2 term.

Proof Since $BP\langle n\rangle \wedge CP^\infty \simeq \vee_{i\geq 0}\Sigma^{2i}BP\langle n\rangle$ it suffices to show that the Adams spectral sequence converging to $BP\langle n\rangle_*(\wedge_t BZ/p)$ collapses to its E_2 term. The following proof of this result was shown to me by Paul Goerss and is heavily dependent on [JW]. Writing M^* for H^*BZ/p the idea is to use their result that:

Theorem 6.16 ([JW])

The Adams spectral sequence

$$\mathrm{Ext}_E(\otimes_t M^*, F_p) \Longrightarrow BP_*(\wedge_t BZ/p)$$

collapses to its E_2 term.

This is useful because we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}_E(\otimes_t M^*, F_p) \arrow[r, Rightarrow] \arrow[d, "R"] & BP_*(\wedge_t BZ/p) \arrow[d] \\
\mathrm{Ext}_{E^n}(\otimes_t M^*, F_p) \arrow[r, Rightarrow] & BP\langle n\rangle_*(\wedge_t BZ/p)
\end{tikzcd}$$

The quotient map $F_p[q_0,\ldots q_n,\ldots] \to F_p[q_0,\ldots,q_n]$ induces an $F_p[q_0,q_1,\ldots]$ module structure on $\mathrm{Ext}_{E^n}(\otimes_t M^*, F_p)$ and R is a module map. To use this information to complete the proof we need.

Lemma 6.17

(a) $R:\mathrm{Ext}^s_E(\otimes_t M^* F_p) \to \mathrm{Ext}^s_{E^n}(\otimes_t M^* F_p)$, is onto, for $s\geq 1$

(b) $\oplus_{s\geq 1}\mathrm{Ext}^s_{E^n}(\otimes_t M^*, F_p)$ is a free $F_p[q_n]$ module

(c) $q_n \cdot \mathrm{Ext}^0_{E^n}(\otimes_t M^*, F_p) \subset \mathrm{Im}\{q_n \cdot R:\mathrm{Ext}^0_E(\otimes_t M^*, F_p) \to \mathrm{Ext}^1_{E^n}(\otimes_t M^*, F_p)\}$

Assuming this lemma for the moment, suppose that $E_2 = E_r$ in the Adams spectral sequence

$$\mathrm{Ext}_{E^n}(\otimes_t M^*, F_p) \Longrightarrow BP\langle n\rangle_*(\wedge_t BZ/p).$$

Take any $x\in\mathrm{Ext}^s_{E^n}(\otimes_t M^*, F_p)$ with $s\geq 1$. Then by 6.17(a) there is some $y\in\mathrm{Ext}_E(\otimes_t M^*, BZ/p)$ such that $Ry=x$, so that $d_r x = d_r Ry = Rd_r y = 0$ by theorem 6.16. Now take any $x\in\mathrm{Ext}^0_{E^n}(\otimes_t M^*, F_p)$. By 6.17(c) there is some $y\in\mathrm{Ext}^0_E(\otimes_t M^*, F_p)$ such that $q_n x = q_n Ry$. Hence

$$q_n d_r x = d_r q_n x = d_r q_n x - q_n R d_r y = d_r(q_n x - q_n Ry) = d_r 0 = 0.$$

Thus 6.17(b) implies that $d_r x = 0$.

So it remains to prove lemma 6.17. Consider the following F_p vector spaces, M_* is H_*BZ/p, B_* is the odd degree part of M_*, and L_{s*} is the even degree part of M_* in degrees less than $2p^s$. Lemma 6.17 is an immediate corollary of the following two lemmas.

Lemma 6.18 ([JW])

$\mathrm{Ext}_{E_s}(\otimes_t M^*, F_p) = \oplus J_1 \otimes \ldots \otimes J_t \otimes F_p[q_m, q_{m+1}, \ldots]$ as F_p vector spaces where the sum is over all sequences $J_1\otimes\ldots\otimes J_t$ such that each J_i is either B_* or L_{k*} where $k-s$ is the number of J_j, $j<i$ which are B_* and where $m-s$ is the number of J_j, $j\leq t$ which are B_*.

All the $J_1 \otimes \dots \otimes J_t$ are in $Ext^{*}_{E_s}(\otimes_t M^*, F_p)$

Lemma 6.19

(a) There is an isomorphism of $F_p[q_n]$ modules

$$Ext_{E_s^n}(\otimes_t M^*, F_p) \cong W \oplus \bigoplus_{(J_1,\dots,J_t)} J_1 \otimes \dots \otimes J_t \otimes F_p[q_{m-1}, \dots, q_n]$$

where W and $J_1 \otimes \dots \otimes J_t$ are in Ext^0 and have trivial $F_p[q_n]$ structure. J_i is either B_* of L_{k*} where $k-s$ is the number of J_j, $j<i$ which are B_* and where $m-s$ is the number of J_j, $j \le t$ which are B_*. Only sequences $J_1 \otimes \dots \otimes J_t$ which have $m \le n$ are allowed.

(b) The reduction map $R: Ext_{E_s}(\otimes_t M^*, F_p) \to Ext_{E_s^n}(\otimes_t M^*, F_p)$ is computed as follows. If $m \le n$ then R is the composite

$$J_1 \otimes \dots \otimes J_t \otimes F_p[q_m, \dots] \to J_1 \otimes \dots \otimes J_t \otimes F_p[q_m, \dots, q_n] \subset Ext_{E_s^n}(\otimes_t M^*, F_p)$$

If $m>n$ then R is

$$J_1 \otimes \dots \otimes J_t \otimes F_p[q_m, \dots] \to J_1 \otimes \dots \otimes J_t \subset W \subset Ext_{E_s^n}(\otimes_t M_*, F_p)$$

Proof The proof is by induction on t, so assume it is true for t in order to do it for $t+1$.

Let L_s^* be the F_p dual to L_{s*}. Define $M_s{}^*$ by

$$0 \to L_s{}^* \to M^* \to M_s{}^* \to 0$$

This splits as E_s and E_s^n modules. Since $L_s{}^*$ is a trivial E_s and E_s^n module there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}_{E_s}(\otimes_{t+1}M^*,F_p) \arrow[r, dashed, "\cong"] \arrow[d, "R"] & L_{s*}\otimes\mathrm{Ext}_{E_s}(\otimes_t M^*,F_p)\otimes\mathrm{Ext}_{E_s}(M_s*\otimes_t M^*,F_p) \arrow[d, "\mathrm{Id}\otimes R\otimes R"] \\
\mathrm{Ext}_{E_s^n}(\otimes_{t+1}M^*,F_p) \arrow[r, "\cong"] & L_{s*}\otimes\mathrm{Ext}_{E_s^n}(\otimes_t M^*,F_p)\otimes\mathrm{Ext}_{E_s^n}(M_s*\otimes_t M^*,F_p)
\end{tikzcd}$$

The isomorphisms are $F_p[q_s,\dots]$ and $F_p[q_s,\dots,q_n]$ module maps respectively. The left summand is computed by the induction hypothesis. The right summand is dealt with by proving that the following two diagrams commute and that the indicated isomorphisms in them are isomorphisms.

$$\begin{tikzcd}
\mathrm{Ext}_{E_s}(M_s*\otimes_t M^*,F_p) \arrow[r, "\cong"] \arrow[d, "R"] & B_*\otimes\mathrm{Ext}_{E_{s+1}}(\otimes_t M^*,F_p) \arrow[d, "R"] \\
\mathrm{Ext}_{E_s^n}(M_s*\otimes_t M^*,F_p) \arrow[r, "\cong"] & B_*\otimes\mathrm{Ext}_{E_{s+1}^n}(\otimes_t M^*,F_p)
\end{tikzcd} \qquad \text{(I) } s\le n$$

$$\begin{tikzcd}
\mathrm{Ext}_{E_s}(M_s*\otimes_t M^*,F_p) \arrow[r, "\cong"] \arrow[d, "R"] & B_*\otimes\mathrm{Ext}_{E_{s+1}}(\otimes_t M^*,F_p) \arrow[d] \\
\mathrm{Ext}_{E_s^n}(M_s*\otimes_t M^*,F_p) \arrow[r, "\cong"] & M_{s*}\otimes_t M_*
\end{tikzcd} \qquad \text{(II) } s>n$$

The top isomorphism of both diagrams is given in [JW] and is an isomorphism of $F_p[q_{s+1},q_{s+2},\dots]$ modules. In diagram (I) the bottom homomorphism will be an isomorphism of $F_p[q_{s+1},\dots,q_n]$ modules. If $n=s$ then this is an isomorphism of $F_p[q_n]$ modules because both will be trivial modules. By convention $E_s^n=F_p$ when $n<s$ so that in diagram (II) the bottom map is an isomorphism of trivial $F_p[q_n]$ modules.

Lemma 6.19 follows from (I) and (II) and the induction hypothesis. We only prove (I) since the proof of (II) is similar. To prove (I), filter $M_s{}^*$ by

$$F_rM_s{}^* = \{x \mid |x| \text{ odd and } \geq r \text{ or } |x| \text{ even and } \geq r+2p^s-1\}$$

If $\bar{M}_s{}^*$ is the associated graded object, then as an E_s or E_s^n module

$$\bar{M}_s{}^* = E[Q_s]\otimes B^*$$

Applying $\mathrm{Ext}_{E_s}(\ \ ,F_p)$ and $\mathrm{Ext}_{E_s^n}(\ \ ,F_p)$ to the short exact sequences

$$0 \to F_rM_s{}^*\otimes_tM^* \to F_{r-1}M_s{}^*\otimes_tM^* \to F_rM_s{}^*/F_{r-1}M_s{}^*\otimes_tM^* \to 0$$

gives a diagram of spectral sequences

$$\begin{array}{ccc} \mathrm{Ext}_{E_s}(\bar{M}_s{}^*\otimes_tM^*,F_p) & \Longrightarrow & \mathrm{Ext}_{E_s}(M_s{}^*\otimes_tM^*,F_p) \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{E_s^n}(\bar{M}_s{}^*\otimes_tM^*,F_p) & \Longrightarrow & \mathrm{Ext}_{E_s}(M_s{}^*\otimes_tM^*,F_p) \end{array} \qquad \text{(III)}$$

From the sequences

$$0 \to E[Q_s] \to E_s \to E_{s+1} \to 0$$

$$0 \to E[Q_s] \to E_s^n \to E_{s+1}^n \to 0$$

we get the diagram of change of rings spectral sequences

$$\begin{array}{ccc} \mathrm{Ext}_{E_{s+1}}(\mathrm{Tor}^{E[Q_s]}(\bar{M}_s{}^*\otimes_tM^*,F_p),F_p) & \Longrightarrow & \mathrm{Ext}_{E_s}(\bar{M}_s{}^*\otimes_tM^*,F_p) \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{E_{s+1}^n}(\mathrm{Tor}^{E[Q_s]}(\bar{M}_s{}^*\otimes_tM^*,F_p),F_p) & \Longrightarrow & \mathrm{Ext}_{E_s^n}(\bar{M}_s{}^*\otimes_tM^*,F_p) \end{array} \qquad \text{(IV)}$$

Since $\bar{M}_s{}^*\otimes_tM^* = E[Q_s]\otimes B^*\otimes_tM^*$ is $E[Q_s]$ free,

$$\mathrm{Tor}^{E[Q_s]}(\bar{M}_s{}^*\otimes_t M^*, F_p) = B^*\otimes_t M^*$$

Thus (IV) gives a commutative diagram of isomorphisms where the top is an $F_p[q_{s+1},\ldots]$ module map and the bottom is an $F_p[q_{s+1},\ldots,q_n]$ (or $F_p[q_n]$ if $s=n$) module map.

$$\begin{tikzcd}
B^*\otimes \mathrm{Ext}_{E_{s+1}}(\otimes_t M^*, F_p) \arrow[r, "\approx"] \arrow[d] & \mathrm{Ext}_{E_s}(\bar{M}_s \otimes_t M^*, F_p) \arrow[d] \\
B^*\otimes \mathrm{Ext}_{E^n_{s+1}}(\otimes_t M^*, F_p) \arrow[r, "\approx"] & \mathrm{Ext}_{E^n_s}(\bar{M}_s \otimes_t M^*, F_p)
\end{tikzcd}$$

The only thing left to do is show (III) collapses.

$$\mathrm{Ext}_{E_s}(\bar{M}_s\otimes_t M^*, F_p) \Longrightarrow \mathrm{Ext}_{E_s}(M_s{}^*\otimes_t M^*, F_p)$$

collapses by 6.14 of [JW]. Now take $x\otimes y$, an arbitrary element of $B^*\otimes \mathrm{Ext}^p_{E^n_{s+1}}(\otimes_t M^*, F_p)$. If $p\geq 1$, then, by our induction hypothesis, there is some $z\in \mathrm{Ext}^p_{E_{s+1}}(\otimes_t M^*, F_p)$ such that $Rz=y$. So $d_r(x\otimes y) = d_r(x\otimes Rz) = 0$. If $p=0$ our induction hypothesis tells us that there is some $z\in \mathrm{Ext}^0_{E_{s+1}}(\otimes_t M^*, F_p)$ such that $q_n Rz = q_n y$. Then $q_n d_r(x\otimes y) = d_r(x\otimes q_n y) = d_r(x\otimes q_n y - x\otimes q_n Rz) = 0$. Our induction hypothesis also implies that multiplication by q_n is nontrivial on $\mathrm{Ext}^r_{E^n_s}(\otimes_t M^*, F_p)$ so $d_r(x\otimes y) = 0$. Q.E.D.

Finally we are in a position to prove the existence of the adaptive complexes we need.

<u>Proof of 6.11</u> Let $R = \{\chi P^i \mid Q_2 Q_1 Q_0 P^I \neq 0,\ I \text{ admissible},\ i_1 \geq k\}$.

If $\epsilon: A \to M_2(k)$ is the projection map then $\epsilon(R)$ forms a basis for $M_2(k)$. Let e,s and t be as in 6.14. Let X_I be a finite skeleton of $C(s,t+3)$ containing the $2p(k+1)+|Q_0|+|Q_1|+|Q_2|$ skeleton and such that $HZ^{\wedge}_{p*}X_I$ is an F_p vector space so that $HZ^{\wedge}_{p*}X_I \to HZ/p_*X_I$ is a monomorphism. Let $j=2pk-2pi_1$ and $Z_I = \Sigma^j X_I$. Let $c \in H^{j+e+3}Z_I$ be the generator so that by lemma 6.14

$$prP^I c \neq 0 \text{ in } H^*Z_I/\bar{E}^n \cdot H^*Z_I.$$

If $h_I = (prP^I c)^* \in H_{2pk+n+1}Z_I$ then $(P^I)^* h_I \neq 0$.

Theorem 6.15 guarantees that h_I is in the image of the reduction $BP\langle 2\rangle_* Z_I \to H_* Z_I$. Let h_I' be any preimage of h_I and we have our adaptive complex for $BP\langle 2\rangle$.

The fact that $P^I c$ generates a free E^2 module in H^*Z_I means that h_I' will generate a module with trivial p, v_1 and v_2 actions. This means that if we split $BP\langle 2\rangle \wedge Z_I \simeq B \vee KV$ where KV is the Eilenberg-MacLane spectrum for some graded vector space V and H^*B has no free summands over A then $h_I': S \to B \vee KV$ is given by $*\vee f$ for some $f: S \to KV$. Since for any spectrum X, $[KV, X] \simeq \mathrm{Hom}_A(A_*, H_*X)$ (the appropriate Adams spectral sequence collapses) we notice that the composite $S^j \circ h_I$ is nonzero if and only if the induced map in homology is. Consider

$$\begin{array}{ccccc}
S^m & \xrightarrow{h_I'} & B \vee KV & \xrightarrow{S^j \wedge Id} & \Sigma^{qj} BP\langle 1\rangle \wedge Z_I \\
 & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\
 & & HZ/p \wedge Z_I & \xrightarrow{\chi P^j \wedge Id} & \Sigma^{qj} HZ/p \wedge Z_I
\end{array}$$

For dimensional reasons since Z_I is a space, if $j>k$, $\chi P^j \circ \rho \circ \bar{h}_I' = *$.

Since the reduction from $BP\langle 1\rangle$ to HZ/p is a monomorphism in homology we find that $S^j \circ h_I'$ is zero in homology. Hence $S^j \circ h_I' = *$ when $j>k$ so $d_{0*} h_I' = 0$. Q.E.D.

6.3 <u>Construction of $BP\langle 2\rangle^k$</u>

Let

$$K_0 \xrightarrow{d_0} \Sigma K_1 \xrightarrow{d_1} \Sigma^2 K_2 \longrightarrow \cdots$$

be the sequence of spectra constructed in section 6.1. We will construct cofiber sequences

$$K_{q+1} \xrightarrow{i_{q+1}} X_{q+1} \xrightarrow{p_q} X_q \xrightarrow{e_q} \Sigma K_{q+1}$$

with $e_q i_q = d_q$, $X_0 = K_0^2 = BP\langle 2\rangle$ and $e_0 = d_0$. Recall that in section 6.2 we have constructed an adaptive complex, T, for $M_2(k)$ with a map $h: T \to K_0^2$ so that

$$H^* K_1 \xrightarrow{d_0^*} H^* K_0 \xrightarrow{h^*} H^* T$$

is exact with liftings $h_q: T \to X_q$ for $q \leq t+1$ so that

$$\begin{tikzcd} T \arrow[r, "h_{q+1}"] \arrow[d] & X_{q+1} \arrow[d, "p_q"] \\ T \arrow[r, "h_q"] & X_q \end{tikzcd}$$

commutes. Before doing this we need one last lemma which encapsulates the role the adaptive complex plays.

<u>Lemma 6.20</u>

Given the cofiber sequences (I) for $q \leq t$ and liftings of h to X_{t+1} satisfying (II), $\ker i_q^* \cap \ker h_q^* = 0$. for $q \leq t$.

<u>Proof</u> The proof is by induction on q. There are two base cases. If q=0 the result follows since $i_0^* = 0$. If q=1, take v ker i_1^*. Then $v = p_1^* w$ for some $w \in H^* X_0$. Now ker p_1^* = Im e_0^* implies that either v=0 in which case we are done or else $w \notin$ Im e_0^* = Im d_0^* = ker h_0^* by the hypothesis of the lemma. Thus $0 \neq h_0^* w = h_0^* p_1^* w = h_1^* v$ since h_1 is a lift of h_0. Hence ker $i_1^* \cap$ ker $h_1^* = 0$. Now assume

ker $i_q^* \cap$ ker $h_q^* = 0$ for $q \le s < t$. Take $v \in$ ker i_{s+1}^*. Then $v = p_s^* w$ for some $w \in H^* X_s$. Now $i_s^* w \in$ ker d_{s-1}^* = Im d_s^* so choose $x \in H^* K_{s+1}$ such that $d_s^* x = i_s^* w$. Let $w' = w - e_s^* x$. Then $w' \in$ ker i_s^* so by our induction hypothesis $h_s^* w' \neq 0$. But

$$h_s^* w' = h_{s+1}^* \circ p_s^* (w - e_s^* x) = h_{s+1}^* v. \qquad \text{Q.E.D.}$$

Recall the sequence

$$BP\langle 2\rangle = K_0^2 \xrightarrow{d_0} \Sigma K_1^2 \xrightarrow{d_1} \Sigma^{-1} K_2^2 \quad \ldots$$

of proposition 6.10. Theorem 6.3 is an immediate corollary of

<u>Theorem 6.21</u>

There exist spectra X_q, $q \ge 0$ and maps $e_q : X_q \to \Sigma K_{q+1}^2$ so that

(a) $X_0 = K_0^2 = BP\langle 2\rangle$ $e_0 = d_0$

(b) There are cofiber sequences

$$K_{q+1}^2 \xrightarrow{i_{q+1}} X_{q+1} \xrightarrow{p_q} X_q \xrightarrow{e_q} \Sigma K_{q+1}^2$$

(c) $e_q i_q = d_q$, $i_0 = \mathrm{Id}$

(d) For any CW complex the induced map of homology theories

$$e_{q*}:(X_q)_m Z \to (K^2_{q+1})_m Z$$

is zero for $m \le 2p(k+1)-1$ and $q \ge 2$.

To get theorem 6.3 from this just take $BP\langle 2\rangle^k = \operatorname{holim} X_q$

Proof of 6.21 Let

$$\cdots \to Y_q \xrightarrow{p_{q-1}} Y_{q-1} \to \cdots \to Y_1 \to Y_0 = HZ/p$$

be the tower whose homotopy inverse limit is $B(k)$ as in theorem 6.6. We have the cofiber sequences

$$K_{q+1} \xrightarrow{i_{q+1}} Y_{q+1} \xrightarrow{p_q} Y_q \xrightarrow{e_q} K_{q+1}$$

and maps as in proposition 6.10. Let (Z_k, h_k') be the complex adapted to $M_2(k)$ constructed in 6.11. Set $T = \Sigma^{2pk+3} DZ_k$ and let

$$h: T \to BP\langle 2\rangle$$

be dual to h_k'.

We proceed by induction on q with the hypothesis:

H(t): For $q \le t+1$ there are spectra X_q and maps e_q for $q \le t$ so that 6.21 (a) - (d) hold. Additionally there are maps

$$\Theta_q': X_q \to Y_q \qquad q \le t+1$$

so that for $q \le t$

$$\begin{array}{ccccccc} K^2_{q+1} & \xrightarrow{i_{q+1}} & X_{q+1} & \xrightarrow{p_q} & X_q & \xrightarrow{e_q} & \Sigma K^2_{q+1} \\ \downarrow{\scriptstyle \Theta_{q+1}} & & \downarrow{\scriptstyle \Theta'_{q+1}} & & \downarrow{\scriptstyle \Theta'_q} & & \downarrow{\scriptstyle \Theta_{q+1}} \\ K_{q+1} & \xrightarrow{i_{q+1}} & Y_{q+1} & \xrightarrow{p_q} & Y_q & \xrightarrow{e_q} & \Sigma K_{q+1} \end{array} \qquad (6.22)$$

is a commutative diagram of cofibration sequences. Finally h lifts to X_{t+1}.

Now X_0 is $BP\langle 2\rangle$ and e_0 is $d_0 : BP\langle 2\rangle \to \vee_{j\geq k}\Sigma^{qj}BP\langle 1\rangle$. X_1 is defined to be the fiber of e_0 and setting i_0 to be the identity gives $e_0 i_0 = d_0$. So $H(0)$ is true.

$H(1)$ needs to be dealt with in a special manner. Define e_1 to fill in the following diagram of cofibrations, where the top square commutes because $d_1 \circ d_0 = 0$.

$$\begin{tikzcd}
\Sigma BP\langle 2\rangle \arrow[r] \arrow[d, "d_0"] & * \arrow[d] \\
\Sigma \vee_{i\geq k}\Sigma^{qi}BP\langle 1\rangle \arrow[r, "d_1"] \arrow[d, "i_1"] & \Sigma \vee_{i,j}\Sigma^{q(i+j)}BP\langle 0\rangle \arrow[d, equal] \\
X_1 \arrow[r, dashed, "e_1"] & \Sigma \vee_{i,j}\Sigma^{q(i+j)}BP\langle 0\rangle
\end{tikzcd}$$

So by definition $e_1 i_1 = d_1$. To get $e_1\Theta_1' = \Theta_2 e_1$, notice that

$$\Delta = (e_1\Theta_1' - \Theta_2 e_1)i_1 = e_1\Theta_1' i_1 - \Theta_2 d_1 = d_1\Theta_1 - \Theta_2 d_1 = 0$$

Thus the top square in the following diagram commutes defining g.

$$\begin{tikzcd}
\Sigma \vee_{i\geq k}\Sigma^{qi}BP\langle 1\rangle \arrow[r] \arrow[d, "i_1"] & * \arrow[d] \\
X_1 \arrow[r, "e_1\Theta_1' - \Theta_2 e_1"] \arrow[d, "p_0"] & K_2 \arrow[d, equal] \\
BP\langle 2\rangle \arrow[r, "g"] & K_2
\end{tikzcd}$$

If $\rho : BP\langle 2\rangle \to H$ is the reduction map, then since $\rho^* : H^*H \to H^*BP\langle 2\rangle$

is surjective and K_2 is a generalized Eilenberg MacLane spectrum, g factors as

$$BP\langle 2\rangle \xrightarrow{\rho=\Theta_0'} H=Y_0 \xrightarrow{g'} K_2.$$

Define

$$e_1' = e_1 - g' \circ p_0 : Y_1 \to K_2$$

Notice that in the tower defining $B(pk+1)$ given in theorem 6.6, e_1 can be replaced by e_1' since

$$e_1' i_1 = (e_1 - g' \circ p_0) i_1 = e_1 i_1 = d_1.$$

This replacement may require adjustment of the higher e_i's but this can be done as in the original proof. So

$$e_1' \Theta_1' - \Theta_2 e_1 = \Delta - g' p_0 \Theta_1 = \Delta - g' \Theta_0' p_0 = \Delta - \Delta = 0.$$

To get the lifting of h_1 notice that $\Theta_1' \circ h_1$ lifts so $\Theta_2 \circ e_1 \circ h_1 = e_1 \circ \Theta_1' \circ h_1 = 0$. T was chosen so that $BP\langle 0\rangle * T \to HZ/p * T$ is a monomorphism hence $e_1 \circ h_1 = 0$ and h_1 lifts to $h_2 : T \to X_2$ completing the verification of H(2).

Now assume H(t-1) with $t \geq 2$. Since $t+1 \geq 3$ there is a map $s_{t+1} : K_{t+1} \to K_{t+1}^2$ such that $s_{t+1}\Theta_{t+1}$ is the identity. Set

$$e_t = s_{t+1} e_t \Theta_t' : X_t \to \Sigma K_{t+1}^2$$

and let X_{t+1} be the fiber of e_t. Then by our induction hypothesis

$$e_t i_t = s_{t+1} e_t \Theta_t' i_t = s_{t+1} e_t i_t \Theta_t = s_{t+1} d_t \Theta_t = s_{t+1} \Theta_{t+1} d_t = d_t$$

which gives 6.21(c). To lift h_t to X_{t+1} notice that $\Theta_t' h_t$ is a lifting of $1_* h$. Thus by 6.7 $e_t \Theta_t' h_t = 0$ so $0 = s_{t+1} e_t \Theta_t' h_t = e_t h_t$ and h_t lifts. To get the diagram 6.22 we need to show

$$\begin{tikzcd} X_t \arrow[r, "e_t"] \arrow[d, "\Theta'_t"] & \Sigma K^2_{t+1} \arrow[d, "\Theta_{t+1}"] \\ Y_t \arrow[r, "e_t"] & \Sigma K_{t+1} \end{tikzcd}$$

commutes and then simply define Θ'_{t+1} to satisfy 6.22. Since K_{t+1} is an Eilenberg-MacLane spectrum it suffices to show $\Theta_{t+1}e_t = e_t\Theta'_t$ in cohomology. Lemma 6.20 indicates that it is sufficient to show $i_t^*e_t^*\Theta_{t+1}^* = i_t^*\Theta_t'^*e_t^*$ and $h_t^*e_t^*\Theta_{t+1}^* = h_t^*\Theta_t'^*e_t^*$. However

$$\Theta_{t+1}e_ti_t - e_t\Theta'_ti_t = \Theta_{t+1}d_t - e_ti_t\Theta_t = \Theta_{t+1}d_t - d_t\Theta_t = 0$$

and an earlier argument showed $e_th_t = 0 = e_t\Theta_t'h_t$.

Finally 6.21(d) follows from 6.6 since

$$(\Theta_{t+1})_*:(K^2_{t+1})_*Z \to (K_{t+1})_*Z$$

is injective. Q.E.D.

CHAPTER 7. A SPLITTING OF $\mathrm{HOLIM}_k\ BP\langle 2\rangle \wedge P_{-k}$

Let

$$\to P_{-k-1} \to P_{-k} \to \cdots \to P_0 \tag{7.1}$$

be the inverse system of spectra constructed in [L] and [DM]. A corrected conjecture of [DM] is that there is an equivalence of spectra

$$\operatorname{holim}_k P_{-k} \wedge BP\langle n\rangle \simeq \Pi_{k\in Z}\, \Sigma^{2k-1} BP\langle n-1\rangle^{\wedge}$$

where $E^{\wedge}$ denotes the 2-adic completion of E. This was proved for n=1 in [DM]. This chapter will prove the case n=2 and a generalization to all primes and is the result of joint work with D.M. Davis, D.C. Johnson, M. Mahowald, and S. Wegmann.

Consider the following constructions of [D] and [MP]. Let p be any prime and q=2(p-1). If $\beta' : \Sigma_p \to U(p-1)$ is the reduced standard representation of Σ_p, let β be the induced complex (p-1)-plane bundle over $B\Sigma_p$. When restricted to BZ/p, β has sphere bundle equivalent to that of $(p-1)\lambda$, where λ is the canonical line bundle. Thus for any integer k, there is a map

$$T(k(p-1)\lambda) \to T(k\beta) \tag{7.2}$$

where T( ) denotes the Thom spectrum. The spectra in 7.2 are denoted L_{qk} and P_{qk} respectively. Notice that L_{qk} has one cell of each dimension $\geq qk$, while P_{qk} has one cell of each dimension $\geq qk$ which is congruent to 0 or -1 mod q. H^*L_k is the submodule of classes of degree $\geq k$ in $\Delta = E[x]\otimes F_p[y^{\pm 1}]$ where $|x|=1$, $|y|=2$, $\beta x=-y$, $P^a y^b = \binom{b}{a} y^{b+a(p-1)}$. By 1.1 of [D] appropriate skeleta of L_{qk} and P_{qk} are stably equivalent to stunted lens spaces and stunted $B\Sigma_p$'s respectively. Thus there are compatible collapse maps

$$\begin{tikzcd} L_{qk} \arrow[r, "c"] \arrow[d] & L_{q(k+1)} \arrow[d] \\ P_{qk} \arrow[r, "c"] & P_{q(k+1)} \end{tikzcd}$$

and by collapsing intermediate cells we define

$$L_n = L_{qk}/L_{qk}^{(n-1)} \qquad \text{if } qk<n$$

and

$$P_{q(k+1)-1} = P_{qk}/S^{qk}$$

to obtain inverse systems,

$$\cdots \to L_{-q(k+1)} \to \cdots \to L_{-qk-1} \to L_{-qk} \quad \cdots \to L_0$$

$$\cdots \to P_{-q(k+1)} \to P_{-qk-1} \to P_{-qk} \to \cdots \to P_0.$$

If $p=2$ then $L_n = P_n$ and they agree with the spectra of 7.1.

<u>Theorem 7.3</u>

If p is any prime and $q=2(p-1)$ then

$$\operatorname{holim}_k P_{-k} \wedge BP\langle 2\rangle \simeq \prod_{k\in Z} \Sigma^{qk-1} BP\langle 1\rangle\hat{}$$

and

$$\operatorname{holim}_k L_{-k} \wedge BP\langle 2\rangle \simeq \prod_{k\in Z} \Sigma^{2k-1} BP\langle 1\rangle\hat{}.$$

In order to state some results of which theorem 7.3 will be a corollary we need to review the following material about p-series and formal group laws. Since $BP\langle 2\rangle$ has a CP^∞ orientation in the sense of Part II of [A1], $BP\langle 2\rangle_* CP^\infty$ is a free $BP\langle 2\rangle_*$ module on generators $\beta_i \in BP\langle 2\rangle_{2i}CP^\infty$, $i\geq 0$ and $BP\langle 2\rangle^*CP^\infty$ is the ring of formal power series $BP\langle 2\rangle_*[[x]]$ where x generates $BP\langle 2\rangle^2CP^\infty$. If $m:CP^\infty \times CP^\infty \to CP^\infty$ is the H-space multiplication for CP^∞ then m^*x is a formal power series in two variables, $m^*x = \Sigma_{i,j}\, a_{ij}x_1{}^i x_2{}^j$ with $a_{ij} \in BP\langle 2\rangle_*$. Then the formal group law for $BP\langle 2\rangle$ is given by the formal sum

$$x +_F y = \Sigma_{i,j}\, a_{ij}x^i y^j$$

Let $[n](x) = [n-1](x) +_F x$ where $[1](x) = x$. Then $[p](x) = \Sigma_j c_j x^j$ is the p-series. The spectrum BP also has a CP^∞ orientation and hence a formal group law $m^*x = \Sigma_{i,j}\, b_{ij}x_1{}^i x_2{}^j$, $b_{ij} \in BP_*$. The reduction map $\rho:BP \to BP\langle 2\rangle$ induces a homomorphism of formal group laws. In particular if $\rho_*:BP_* \to BP\langle 2\rangle_*$ is reduction mod $(v_3,v_4,\ldots,)$ then $a_{ij} = \rho_*(b_{ij})$ and lots of information about the formal group law for BP can be found in [Wi] and [H] amongst others.

Spectra CP_k for any integer k can be constructed from stunted complex projective spaces, in a manner similar to the construction of their real analogs P_k, either as Thom spectra or using James periodicity.

Theorem 7.4

For all integers k there are cofibrations

$$L_{2k-1} \to CP_k \xrightarrow{q} T_k \to \Sigma L_{2k-1}.$$

$BP\langle 2\rangle_*T_k$ and $BP\langle 2\rangle_*CP_k$ are free $BP\langle 2\rangle_*$ modules on generators $\gamma_i \in BP\langle 2\rangle_{2i}T_k$ and $\beta_i \in BP\langle 2\rangle_{2i}CP_k$, $i\geq k$. If $c_j \in BP\langle 2\rangle_*$ are defined by $[p](x) = \Sigma_j\, c_j x^{1+(p-1)j}$ then $q_*(\beta_i) = \Sigma_j\, c_j \gamma_{i-(p-1)j}$.

<u>Theorem 7.5</u>

For each integer k, there is a map of cofibrations

$$\begin{tikzcd}
CP_k \wedge BP\langle 2\rangle \arrow[r] & T_k \wedge BP\langle 2\rangle \arrow[r] & \Sigma L_{2k-1} \wedge BP\langle 2\rangle \\
\bigvee_{i\geq k+p^2-1} \Sigma^{2i} BP\langle 2\rangle \arrow[r] \arrow[u] & \bigvee_{i\geq k} \Sigma^{2i} BP\langle 2\rangle \arrow[r] \arrow[u] & \bigvee_{i\geq k} \Sigma^{2i} BP\langle 1\rangle \arrow[u, "g"']
\end{tikzcd}$$

In cohomology $g^*(\Sigma xy^{i-1}\otimes 1) = \Sigma_{j\geq 0}(-1)^j p^j G_{2i-qj}$ where G_{2i} generates $H^*\Sigma^{2i}BP\langle 1\rangle$.

<u>Theorem 7.6</u>

(a) For any integer i, there is a map $\Sigma^{2i-1}BP\langle 1\rangle \to \mathrm{holim}_k L_{-k}\wedge BP\langle 2\rangle$ such that for any integer k, the cohomology homomorphism induced by the composite

$$\Sigma^{2i-1}BP\langle 1\rangle \to \mathrm{holim}_k L_{-k}\wedge BP\langle 2\rangle \to L_{2k-1}\wedge BP\langle 2\rangle$$

sends $xy^{i-1+(p-1)j}\otimes 1$ to $(-1)^j p^j G_{2i-1}$.

(b) The maps of (a) induce an equivalence

$$(\bigvee_{i\in Z} \Sigma^{2i-1}BP\langle 1\rangle)^\wedge \to \mathrm{holim}_k L_{-k}\wedge BP\langle 2\rangle.$$

(c) There is an equivalence

$$(\bigvee_{i\in Z} \Sigma^{2i-1}BP\langle 1\rangle)^\wedge \to \Pi_{i\in Z} \Sigma^{2i-1}BP\langle 1\rangle^\wedge.$$

(d) There are equivalences

$$\Pi_{k\in Z} \Sigma^{qk-1}BP\langle 1\rangle^\wedge \leftarrow (\bigvee_{i\in Z} \Sigma^{qi-1}BP\langle 1\rangle)^\wedge \to \mathrm{holim}_k P_{-k}\wedge BP\langle 2\rangle.$$

Remark The p-completion in this theorem is the p-profinite completion of Sullivan [Su] and Margolis [M]. Notice that for spectra of finite type, p-profinite completion coincides with some of the more usual notions of p-completion. However $\vee_{i \in Z} \Sigma^{2i-1} BP\langle 1\rangle$ is not of finite type.

Proof of 7.4 Let H be the Hopf line bundle over CP^∞ and T the Thom spectrum of $\otimes^p H$ over CP^∞ and T^K its 2k skeleton. There is a cofibration

$$L^{2k-1} \xrightarrow{h} CP^{k-1} \xrightarrow{q} T^K \longrightarrow \Sigma L^{2k-1}$$

where h is the canonical map and L^{2k-1} denotes the skeleton of the lens space BZ/p. If q is made skeletal, the the mapping cone $MC(CP^{k-1} \xrightarrow{q} T^{2k-2})$ is ΣL^{2k-2} so that a commutative diagram of cofibrations

$$\begin{array}{ccccccc}
L^{2k-2} & \xrightarrow{h} & CP^{k-1} & \xrightarrow{q} & T^{k-1} & \longrightarrow & \Sigma L^{2k-2} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L^\infty & \xrightarrow{h} & CP^\infty & \xrightarrow{q} & T & \longrightarrow & \Sigma L^\infty \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L_{2k-1} & \xrightarrow{h} & CP^k & \xrightarrow{q} & T_k & \longrightarrow & \Sigma L_{2k-1}
\end{array}$$

is obtained defining T_k when $k>0$. That $BP\langle 2\rangle_*(q)$ corresponds to the p-series is well known (see e.g. [Wi]) and follows from the fact that the classifying map for $\otimes^2 H$ is $CP^\infty \longrightarrow CP^\infty \times CP^\infty \xrightarrow{m} CP^\infty$ which is used to define $x +_F x$ in $BP\langle 2\rangle^* CP^\infty$.

Let A_n denote the J-order of the Hopf bundle over CP^n, the Atiyah-Todd number ([AW]). If $N \equiv 0$ (A_{k-2}) then there are

J-trivializations of the bundles $2N\xi_{2k-3}$ and NH_{k-2} compatible with the natural map between them ([AW],[KMT]). The identification of stunted lens spaces as Thom complexes shows that if $N\equiv 0$ (A_{k-2}) there is a commutative diagram,

$$\begin{tikzcd}
\Sigma^{2N}L_1^{2k-2} \arrow[r, "h"] \arrow[d, "\cong"] & \Sigma^{2N}CP_1^{k-1} \arrow[d, "\cong"] \\
L_{2N+1}^{2N+2k-2} \arrow[r, "h"] & CP_{N+1}^{N+k-1}
\end{tikzcd}$$

where these can be chosen compatibly as N and k are varied. Thus for k negative, T_k can be defined as the spectrum whose $(2k+2M)$-skeleton T_k^{k+M} satisfies

$$\Sigma^{2N}T_k^{k+M} \simeq MC(L_{2k-1+2N}^{2k+2M+2N} \xrightarrow{h} CP_{k+N}^{k+M+N})$$

if $N\equiv 0$ (A_M) and $k+N>0$. Q.E.D.

Now we work toward the proof of 7.5. The generators of $BP\langle 2\rangle_*CP_k$ and $BP\langle 2\rangle_*T_k$ can be used to construct homotopy equivalences

$$\begin{tikzcd}
h_1 : \vee_{i\geq k}\, \Sigma^{2i}BP\langle 2\rangle \arrow[r, "\simeq"] & CP_k \wedge BP\langle 2\rangle \\
S^{2j}\wedge BP\langle 2\rangle \arrow[u] \arrow[r, "\beta_j\wedge Id"] & CP_k\wedge BP\langle 2\rangle\wedge BP\langle 2\rangle \arrow[u, "Id\wedge m"']
\end{tikzcd}$$

and similarly $h_2 : \vee_{i\geq k}\, \Sigma^{2i}BP\langle 2\rangle \longrightarrow T_k\wedge BP\langle 2\rangle$. Taking advantage of the fact that $\vee_{i\geq k}\, \Sigma^{2i}BP\langle 2\rangle \simeq \Pi_{i\geq k}\, \Sigma^{2i}BP\langle 2\rangle$ we use $\Sigma_j c_j\gamma_{i-(p-1)j}$ to construct the following diagram which defines q'

$$\begin{tikzcd}
\vee_{i\geq k}\, \Sigma^{2i}BP\langle 2\rangle \arrow[rr, "q'"] & & \vee_{i\geq k}\, \Sigma^{2i}BP\langle 2\rangle \\
\Sigma^{2i}BP\langle 2\rangle \arrow[u] \arrow[r, "c_j\wedge Id"] & \Sigma^{2i-(p-1)j}BP\langle 2\rangle\wedge BP\langle 2\rangle \arrow[r, "m"] & \Sigma^{2i-2(p-1)j}BP\langle 2\rangle \arrow[u]
\end{tikzcd}$$

Now

$$\begin{tikzcd}
CP_k \wedge BP\langle 2\rangle \arrow[r, "q\wedge 1"] & T_k \wedge BP\langle 2\rangle \\
\bigvee_{i\geq k} \Sigma^{2i} BP\langle 2\rangle \arrow[r, "q'"] \arrow[u, "h_1", "\simeq"'] & \bigvee_{i\geq k} \Sigma^{2i} BP\langle 2\rangle \arrow[u, "h_2", "\simeq"']
\end{tikzcd} \tag{7.7}$$

commutes. Let $\tilde{q}$ denote the restriction of q' to all but the first p^2-1 summands, and C_k the cofibre of $\tilde{q}$. The map g in 7.5 will be the induced map of cofibres once we have shown $C_k \simeq \vee_{i\geq k} \Sigma^{2i}BP\langle 1\rangle$ and will be a $BP\langle 2\rangle$ module map. Because of our theorem 4.5, in order to prove 7.5 it suffices to prove:

<u>Proposition 7.8</u>

$H^*C_k \simeq \oplus_{i\geq k} \Sigma^{2i}A//E^1$ as A modules with generators G_{2j} satisfying the equation of 7.5.

<u>Proof</u> By the uniformity of the homomorphism q_* it suffices to consider k=0. Since h_2 in 7.7 is an equivalence, there is a commutative diagram of cofibrations,

$$\begin{tikzcd}
\bigvee_{i=0}^{p^2-2} \Sigma^{2i+1}BP\langle 2\rangle \arrow[r] & \bigvee_{i=0}^{p^2-2} \Sigma^{2i+1}BP\langle 2\rangle \\
\Sigma L_{-1}\wedge BP\langle 2\rangle \arrow[r] \arrow[u, "k"] & \bigvee_{i\geq 0} \Sigma^{2i+1}BP\langle 2\rangle \arrow[u] \\
C_0 \arrow[r] \arrow[u, "g"] & \bigvee_{i\geq p^2-1} \Sigma^{2i+1}BP\langle 2\rangle . \arrow[u]
\end{tikzcd}$$

The A module $H^*(\Sigma L_{-1}\wedge BP\langle 2\rangle)$ is generated by $\{xy^{j-1}\otimes 1 | j\geq 0\}$ with relations $Q_0(\Sigma xy^{j-1}\otimes 1) = Q_1(\Sigma xy^{j-p}\otimes 1) = Q_2(\Sigma xy^{j-p^2}\otimes 1)$ where terms are ignored (not set = 0) if the superscript of y is less than -1.

Since $k^*(\Sigma^{2i+1}\iota) = \Sigma y^i \otimes 1$ (ι generates $H^*BP\langle 2\rangle$), H^*C_0 is the A module given by generators a_{2i}, $i\geq 0$ and relations $Q_0 a_{2i} = Q_1 a_{2i-q} = Q_2 a_{2i-2p^2+2}$ where $a_{2i} = g^*(\Sigma xy^{i-1}\otimes 1)$ and if $n<0$ then a_n is interpreted to be zero.

An isomorphism Φ of the above presentation of H^*C_0 with $\oplus_{i\geq 0}\, \Sigma^{2i}A//E^1$ is given by

$$\Phi(a_{2i}) = \Sigma_{j=0}^{i/(p-1)} (-1)^j P^j G_{2i-j}.$$

In order to verify that Φ is well defined one uses

$$P^j Q_i - Q_i P^j = Q_{i+1} P^{j-p^i}$$

to show that $\Phi(Q_0 a_{2i}) = \Phi(Q_1 a_{2i-q}) = \Phi(Q_2 a_{2i-2p^2+2})$. That Φ is an isomorphism follows by a counting argument or by checking that $G_{2i} \to \Sigma_j (-1)^j \chi P^j a_{2i-qj}$ is an inverse. Q.E.D.

The proof of theorem 7.6 depends on the following result of [W].

Theorem 7.9 ([5.6;W])

If X is a spectrum of finite type and $\{Y_k\}$ is an inverse system of spectra each of finite type, then there is a spectral sequence converging strongly to $[X, \mathrm{holim}_k Y_k]$ with $E_2 = \mathrm{Ext}_A(\mathrm{colim}_k H^*Y_k, H^*X)$.

Proof of theorem 7.6 Let Δ denote the A module $\mathrm{colim}_k H^*(L_{-k})$, Δ_k the submodule of classes of degree $\geq k$, and Δ^{k-1} the quotient Δ/Δ_k. It follows from 7.9 that there is an Adams spectral sequence with

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\Delta\otimes A//E^2,\Sigma^{2i-1}A//E^1)$$

converging to $[\Sigma^{2i-1}BP<1>,\mathrm{holim}_k L_{-k}\wedge BP<2>]$. The homomorphism sending $xy^{i+(p-1)j-1}\otimes 1$ to $(-1)^j\Sigma^{2i-1}P^j\iota$ and $y^{i+(p-1)j}$ to $(-1)^{j+1}\Sigma^{2i-1}P^j\iota$ gives an element γ_i of $E_2^{0,0}$ which when restricted to $\mathrm{Ext}_A^{0,0}(\Delta_{2k-1}\otimes A//E^2,\Sigma^{2i-1}A//E^1)$ for any k is the cohomology homomorphism induced by restriction to $\Sigma^{2i-1}BP<1>$ of the map $g:C_k \to \Sigma L_{2k-1}\wedge BP<2>$ of 7.5. (To see that this is A linear, one verifies that the analogous morphism $\Delta \to \Sigma^{2i-1}A//E^1$ is E^2 linear hence extends to $A\otimes_{E^2}\Delta \to \Sigma^{2i-1}A//E^1$ and then uses the A isomorphism $A//E^2\otimes\Delta \to A\otimes_{E^2}\Delta$.)

Because $\mathrm{Ext}_A^{s,t}(\Delta^{2k-2}\otimes A//E^2,\Sigma^{2i-1}A//E^1)=\mathrm{Ext}_{E^2}^{s,t}(\Delta^{2k-2},\Sigma^{2i-1}A//E^1)=0$ if $t-s \geq 2k-2i+2s(p^2-1)$, one uses the long exact sequence of Ext groups associated to $0 \to \Delta_{2k-1} \to \Delta \to \Delta^{2k-2} \to 0$ to see that the restriction

$$\rho_{i,k}^{s,t}:\mathrm{Ext}_A^{s,t}(\Delta\otimes A//E^2,\Sigma^{2i-1}A//E^1) \to \mathrm{Ext}_A^{s,t}(\Delta_{2k-1}\otimes A//E^2,\Sigma^{2i-1}A//E^1)$$

is injective in the same range and an isomorphism if $t \geq 2k-2i+(2p^2-1)(s+1)$. Now suppose that $d_r(\gamma_i) \neq 0$ in the Adams spectral sequence converging to $[\Sigma^{2i-1}BP<1>,\mathrm{holim}_k L_{-k}\wedge BP<2>]$. Choose $k < i-(p^2-1)r$. Then $\rho_{i,k}^{r,r-1}$ is injective.

Hence $d_r(\rho_{i,k}^{0,0}(\gamma_i))\neq 0$ in the Adams spectral sequence converging to $[\Sigma^{2i-1}BP\langle 1\rangle, L_{-2k-1}\wedge BP\langle 2\rangle]$ contradicting the assertion that $\rho^{0,0}_{i,k}(\gamma_i)$ is the cohomology homomorphism induced by a map. Hence we get a map $\Sigma^{2i-1}BP\langle 1\rangle \to \mathrm{holim}_k L_{-k}\wedge BP\langle 2\rangle$ whose cohomology effect is as required in 7.6(a).

The maps of 7.6(a) give a map $f_1 : \vee_{i\in Z}\, \Sigma^{2i-1}BP\langle 1\rangle \to \mathrm{holim}_k L_{-k}\wedge BP\langle 2\rangle$. For each k, let ρ_k denote the map $\mathrm{holim}_k L_{-k}\wedge BP\langle 2\rangle \to L_{2k-1}\wedge BP\langle 2\rangle$. These maps are compatible with the maps of the inverse system. Since each $L_{2k-1}\wedge BP\langle 2\rangle$ has finite homotopy groups and hence is p-complete, there are compatible maps

$$(\rho_k f_1)^{\wedge} : (\vee_{i\in Z}\, \Sigma^{2i-1}BP\langle 1\rangle)^{\wedge} \to L_{2k-1}\wedge BP\langle 2\rangle$$

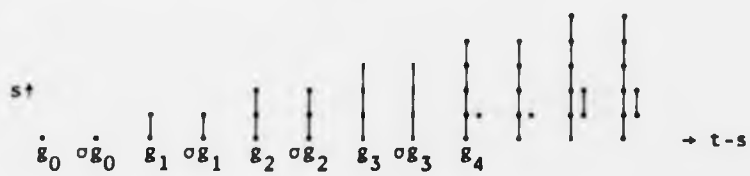
and hence a map

$$f : (\vee_{i\in Z}\, \Sigma^{2i-1}\; BP\langle 1\rangle)^{\wedge} \to \mathrm{holim}_k L_{-k}\wedge BP\langle 2\rangle.$$

The E_2 term of the Adams spectral sequence converging to $\pi_*(L_{2k-1}\wedge BP\langle 2\rangle)$ will be calculated by minimal resolution in lemma 7.11. The result is

$$(F_p[q_0, q_2]\{g_i \mid i \geq 0\}/q_0^{i+1} g_i) \otimes F_p[\sigma]/\sigma^{p-1}$$

where σ, q_0, q_2 and g_i have bidegrees (s,t) equal to (2,0), (1,1), $(1, 2p^2-1)$ and $(0, 2k-1+qi)$ respectively. The diagram below illustrates the situation when p=3 where dots indicate F_p and vertical segments indicate multiplication by q_0.



Since all nonzero elements are in even stems and differentials decrease stem by 1 there are no differentials so that this chart also represents $\pi_*(L_{2k-1}\wedge BP\langle 2\rangle)$ with vertical segments corresponding to multiplication by p.

As noticed earlier in this thesis, the E_2 term of the Adams spectral sequence for computing $\pi_* BP\langle 1\rangle$ is given by the ring $F_p[q_0,q_1]$ where q_0 and q_1 have bidegrees (1,1) and (1,2p-1) respectively. Again there is nothing in the even stems so no differentials and we have a presentation of $\pi_* BP\langle 1\rangle$. The calculation of induced homomorphisms of minimal resolutions in lemma 7.11 will show that the Ext homomorphism induced by a map

$$f:\Sigma^{2k-1+qm+2r}BP\langle 1\rangle \to L_{2k-1}\wedge BP\langle 2\rangle$$

with $0\leq r\leq p-1$ and cohomology effect as in 7.6(a) is

$$f_*\Sigma^{2k-1+qm+2r}q_0{}^j q_1{}^n = \begin{cases}\sigma^r q_0{}^j q_2{}^n g_{m-pn} & \text{if } m\geq pn\\ 0 & \text{else}\end{cases} \qquad (7.10)$$

Let G_m denote the abelian group with generators β_i for $0\leq i\leq m$ and relations $p^{(m-i)(p+1)}\beta_i=0$. Then the Adams spectral sequence calculation of the preceding paragraph shows

$$G_m \cong \pi_{2j-1}(L_{2j-1-q(m(p+1)-1)}\wedge BP\langle 2\rangle)$$

where β_i corresponds to $q_2{}^i g_{(m-i)(p+1)-1}$. Hence the inverse system

$$\cdots \to G_{m+1} \to G_m \to \cdots \to G_1 \qquad (\beta_i \to \beta_i)$$

is cofinal in $\{\pi_{2j-1}(L_{-2k-1}\wedge BP\langle 2\rangle)\,|\,k\to\infty\}$ for any j. Because we are using profinite completion, $\pi_{2j-1}((\vee_{i\in Z}\,\Sigma^{2i-1}BP\langle 1\rangle)^\wedge)$ is a direct product of copies of the p-adic integers with generators α_i, which by 7.10 map to β_i (plus possibly elements of higher filtration) under the homomorphism

$$\pi_{2j-1}((\vee_{i\in Z}\ \Sigma^{2i-1}BP\langle 1\rangle)^{\wedge}) \to \pi_{2j-1}(L_{-2k-1}\wedge BP\langle 2\rangle) = G_m$$

(for appropiate k).

These induce an isomorphism into the inverse limit which proves 7.6(b).

To prove 7.6(c) notice that there are maps

$$(\vee_{i\in Z}\ \Sigma^{2i-1}BP\langle 1\rangle)^{\wedge} \to \Sigma^{2m-1}BP\langle 1\rangle^{\wedge}$$

whose product

$$(\vee_{i\in Z}\ \Sigma^{2i-1}BP\langle 1\rangle)^{\wedge} \to \Pi_{i\in Z}\ \Sigma^{2i-1}BP\langle 1\rangle^{\wedge}$$

is an isomorphism in homotopy.

To prove 7.6(d) notice that the first equivalence follows exactly as in the previous paragraph. To prove the second equivalence we use the composite

$$(\vee_{i\in Z}\ \Sigma^{qi-1}BP\langle 1\rangle)^{\wedge} \to (\vee_{i\in Z}\ \Sigma^{2i-1}BP\langle 1\rangle)^{\wedge} \to \operatorname{holim}_k L_{-k}\wedge BP\langle 2\rangle$$
$$\to \operatorname{holim}_k P_k\wedge BP\langle 2\rangle$$

Calculations similar to those above indicate that this composite is an isomorphism in homotopy. Q.E.D.

<u>Lemma 7.11</u>

(a) The E_2 term of the Adams spectral sequence converging to $\pi_*(L_{2k-1}\wedge BP\langle 2\rangle)$ is

$$(F_p[q_0,q_2]\{g_i \mid i\geq 0\}/q_0^{i+1}g_i)\otimes F_p[\sigma]/\sigma^{p-1}$$

(b) If $f:\Sigma^{2k-1+qm+2r}BP\langle 1\rangle \to L_{2k-1}\wedge BP\langle 2\rangle$ with $0\leq r<p-1$ has the effect in cohomology of sending $xy^{k-1+(q/2)m+r+(p-1)j}\otimes 1$ to

$(-1)^j P^j G_{2k-1+qm+2r}$ then the induced Ext homomorphism is

$$f_*(\Sigma^{2k-1+qm+2r} q_0^{\,j} q_1^{\,n}) = \begin{cases} \sigma^r q_0^{\,j} q_2^{\,n} g_{m-pn} & \text{if } m \geq pn \\ 0 & \text{else} \end{cases}$$

<u>Proof</u> It suffices to consider the case k=0 of $L_{2k-1} \wedge BP\langle 2\rangle$.

$H^*(\Sigma L_{-1} \wedge BP\langle 2\rangle)$ splits over A into a direct sum of (p-1) isomorphic (up to grading) A modules. The bottom summand S has generators $z_i, i \geq 0$ with $|z_i| = qi$ and relations $Q_0 z_i = Q_1 z_{i-1} = Q_2 z_{i-p-1}$ where terms with negative subscripts are ignored. We shall show that a homomorphism $f: S \to \Sigma^{qm} A//E^1$ sending z_{m+j} to $(-1)^j P^j$ for $j \geq 0$ induces $f_*: \mathrm{Ext}_A(\Sigma^{qm} A//E^1, Z/p) \to \mathrm{Ext}_A(S, Z/p)$ as in the case r=0 of the lemma.

A minimal resolution $(\oplus C_i, d)$ of $A//E^1$ is $A \otimes F_p[u_0, u_1]$ where $u_0^{\,i} u_1^{\,n}$ has degree $i+(q+1)n$ in C_{i+n} and

$$d(u_0^{\,i} u_1^{\,n}) = Q_0 \otimes u_0^{\,i-1} u_1^{\,n} + Q_1 \otimes u_0^{\,i} u_1^{\,n-1}$$

(terms are ignored if an exponent is negative). A minimal resolution of S is $A \otimes F_p[w_1, w_2, z]$ where $w_1^{\,i} w_2^{\,j} z^k$ has degree $(q+1)i+(qp+q+1)j+qk$ in C_{i+j} and

$$\begin{aligned} d(w_1^{\,j} w_2^{\,j} z^k) = {} & -Q_1 \otimes w_1^{\,i-1} w_2^{\,j} z^k + Q_0 \otimes w_1^{\,i-1} w_2^{\,j} z^{k+1} \\ & -Q_2 \otimes w_1^{\,i} w_2^{\,j-1} z^k + Q_1 \otimes w_1^{\,i} w_2^{\,j-1} z^{k+p}. \end{aligned}$$

Using the relation $P^j Q_i - Q_i P^j = Q_{i+1} P^{j-p^i}$ one verifies that the homomorphism f is covered by the homomorphism of minimal resolutions

$$w_1^{\,i} w_2^{\,j} z^{m-i-pj+k} \mapsto (-1)^k P^k \otimes u_0^{\,i} u_1^{\,j}. \qquad (7.12)$$

Apply Hom(,Z/p) to these minimal resolutions, naming the dual to $u_0{}^i u_1{}^j$ as $q_o{}^i q_1{}^j$ and the dual to $w_1{}^i w_2{}^j z^k$ as $q_0{}^i q_2{}^j g_{k+i}$. This naming is consistent with the Yoneda action of q_0 on these Ext groups. The dual of 7.12 is 7.11(b) Q.E.D.

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